3. Walrasian equilibrium[15]

3.1. **Ownership structure.** Although the production model described in Section 1.3 mentioned owenership, so far we haven't defined it. It is on purpose: we did not need it to talk about feasible allocations, nor about Pareto-optimality.

However, we are going to need it to talk about the optimal behavior of consumers and who gets the profits obtained by firms. Our model assumes that all endowments and technologies are privately owned: Each agent i owns private endowment $\omega_i \in \mathbb{R}^L$and a share $\theta_{ij} \in [0,1]$ of firm j. We assume that

$$\sum_i \omega_i = \omega \text{ and } \sum_i \theta_{ij} = 1 \text{ for each firm} j.$$

It is a private ownership economy: everything belongs to private individuals. No public goods.

3.2. **Prices and demands.**

- Prices: $p \in \mathbb{R}^L$ - price vector. A price is a payment by a consumer to the producer (or to the selling consumer). In principle, a price can be negative, in which case, the good is sold be the consumer to the producer (for instance, labor). (But, a better convention is that the factors are denoted with approrpriate sign).
- Value of bundle x:
$$p \cdot x = \sum_l p^l x^l.$$
Price is a linear functional on the commodity space.
- Budget set
$$B_i(p, w_i) = \{x \in X_i : p \cdot x \leq w_i\}.$$

3.3. **Walrasian equilibrium.**

- Assumptions of the private ownership, competetive model:
  - *Consumer maximization*: Each consumer chooses optimal consumption bundle in her budget set. Consumer's demand
$$x_i^*(p, w) = \arg\max_{x \in X_i : p \cdot x \leq w} u_i(x)$$

[15] $November 1, 2019$.

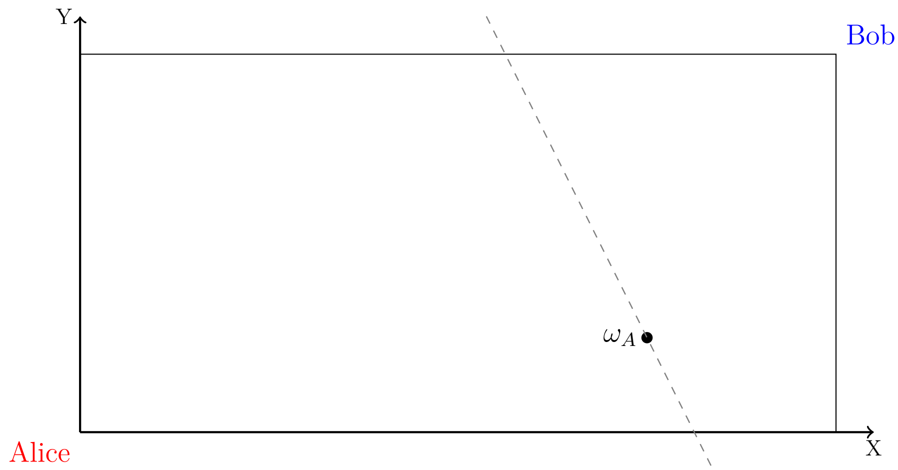

FIGURE 3.1. Walrasian equilibrium in an exchange economy .

– *Firm maximization*: Each firm chooses a bundle in its technology set to maximize the profits. Firm's demand

$$y_j^*(p) = \arg\max_{y \in Y_j} p \cdot y$$

– *Markets clear:* Prices must make sure that the resulting allocation is weakly aggregate feasible.

Definition 3. A Walrasian equilibrium is a allocation $x = (x_1, ..., y_J)$ and a vector of non-negative prices p such that

- (*consumer's optimization*) for each i, $x_i \in x^*(p, w_i)$, where $w_i = p \cdot \omega_i + \sum_j \theta_{ij} (p \cdot y_j)$,
- (*firm's optimization*) for each j, $y_j \in y^*(p)$;
- (*market clearing*) x is feasible.

Pictures:

- Exchange economy with two agents and two goods with convex preferences. Use the picture to talk about relation between Pareto-optimality and Walrasian equilibrium (FWT).
- Production economy with a single agent, technology and convex preferences.

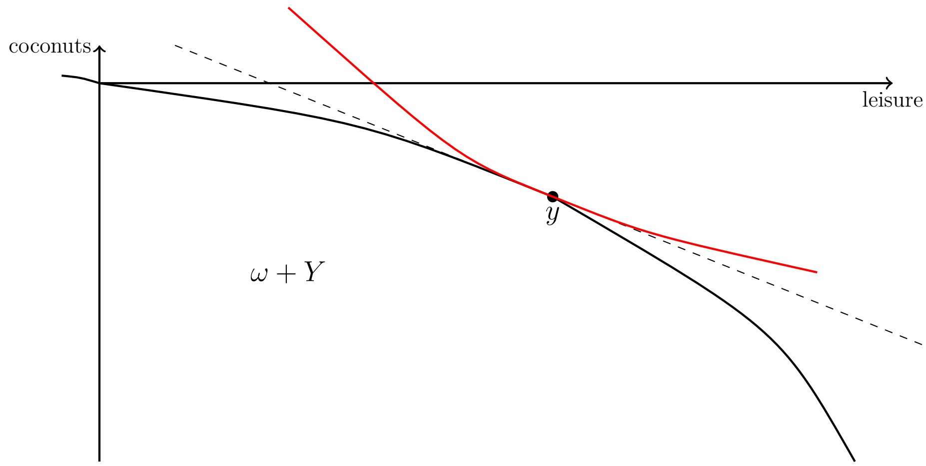

FIGURE 3.2. Walrasian equilibrium in a production economy .

3.3.1. *Underlying assumptions.*

- No consumption externalities (a) the preferences are defined on individual bundles $x_i \in X_i$, not on the allocations $x \in X$; (b) individuals choose their consumption bundles independently.
- No production externalities: firms choose their output bundles independently
- Price taking. Walrasian- auctioneer.
- Costless transactions (though we saw that we can incorporate cost of transactions, like transportation, into the model.)
- Single market. All goods are traded on the same market - meaning that each consumer (and firm) needs to satisfy a single budget constraint.

3.4. **Monotone preferences.** The next Lemma shows that if the preferences of at least one consumer are sufficiently monotone, then the prices must be positive. The result can be quite helpful in eliminating the need to check for "stupid" cases when looking for an equilibrium.

Lemma 2. *Suppose that (x, p) is a Walrasian Equilibrium.*

(1) *If at least one consumer has strongly monotone preferences, then all prices are strictly positive, $p \gg 0$, and*

(2) *If at least one consumer has strictly monotone preferences, then all prices are positive,* $p \geq 0$*.*

(3) *If prices are strictly positive and preferences are locally non-satiated, then allocation* x *satisfies strict aggregate feasibility.*

Proof. 1. If some price $p_l \leq 0$ is non-positive, then the consumer with strongly monotone preferences is not optimizing by choosing a finite amount of good x_i^l.

2. If at least one price is negative, $p_l < 0$ for some l, then the consumer with strictly monotone preferences is not optimizing by choosing bounded consumption bundle. The consumer can choose arbitrarily large amount of good l and use the fact that the price of it is negative to subsidize the consumption of arbitrarily large amounts of all other goods.

3. Locally non-satiated preferences and strictly positive prices imply that the Walras Law holds:

$$p \cdot x_i = p \cdot \omega_i + \sum_j \theta_{ij} p \cdot y_j.$$

By adding the equalities across all consumers, we obtain

$$p \cdot \sum x_i = p \cdot \omega + p \sum_j \cdot y_j.$$

Hence, if the allocation satisfies weak aggregate feasibility, and the prices are strictly positive, it must be that

$$\sum x_i = \omega + \sum_j \cdot y_j.$$

□

3.5. **Examples.** Some helpful tricks to find Walrasian equilibrium

- If at least one agent preferences are strongly monotone, prices must be positive.
- With strictly positive prices and locally non-satiated preferences for consumer i, the Walras Law holds $p \cdot x_i = w_i$..
- With strictly positive prices and locally non-satiated preferences for each consumer, the Walras Law implies aggregate feasibility holds $p \cdot x_i = w_i$ for each i.

- If a firm has constant returns to scale, the equilbrium prices must give 0 profits for any such a firm (or more precisely, any firm that has positive level of activity).
- Scalling prices by a strictly positive factor does not affect the equilibrium. Hence, we can normalize at least one price (at least when it is positive).

3.5.1. *Robinson Crusoe economy.*

Example 12. Suppose that $|I| = |J| = 1, |L| = 2$. There are two good - coconuts and leisure, one consumer - Robinson, and one firm - Coconut, Inc. Robinson owns the firm, owns L_0 units of leisure, and no coconuts. His utility $u(x, l) = x^{\alpha} l^{1-\alpha}$ from coconuts x and leisure l. The firm production function is $f(\lambda) = \lambda^{\beta}$, where λ is the amount of labor (i.e., leisure given to the firm). Assume $\alpha, \beta \in (0, 1)$. Build a model. Find WE. Is it Pareto-optimal?

Two goods: coconuts x and leisure l.

One consumer: Robinson. $X_R = R_+ \times [0, L_0]$. . Endowment $\omega_R = (0, L_0)$. One firm: Coconut, Inc. Technology converts leisure into coconuts: $Y = \left\{(x, -l) : 0 \leq x \leq l^{\beta}, l \geq 0\right\}$. Robinson owns the Coconut, Inc.

Prices (p, w), where $p, w > 0$. We can normalize one price, say $w = 1$.

Consumer demand:

$$p = \frac{\alpha}{1-\alpha} \frac{l}{x},$$

where the budget constraint implies that

$$px = (L_0 - l) + \pi,$$

where π are the equilibrium profits of Coconut, Inc. (More generally, Cobb-Douglas utility implies that the share of spending on each good is proportional to the exponents.)Hence, the consumer's demand for leisure

$$l = (1 - \alpha)(L_0 + \pi).$$

Firm's optimization

$$p = \frac{1}{\beta} \lambda^{1-\beta}.$$

Hence, the profits are equal to

$$\pi = \frac{1}{\beta}\lambda^{1-\beta}\lambda^{\beta} - l = \frac{1-\beta}{\beta}\lambda = \frac{1-\beta}{\beta}\left(\alpha L_0 - (1-\alpha)\pi\right),$$

where we used the fact that $\lambda = L_0 - l = \alpha L_0 - (1-\alpha)\pi$. More algebra leads to

$$\pi = \frac{\alpha(1-\beta)}{\beta + (1-\beta)(1-\alpha)} L_0,$$

and the equilibrium labor is

$$\lambda = \alpha L_0 - (1-\alpha)\frac{\alpha(1-\beta)}{\beta + (1-\beta)(1-\alpha)} L_0 = \frac{\alpha\beta}{\beta + (1-\beta)(1-\alpha)} L_0 = \frac{\alpha\beta}{1-\alpha+\beta\alpha} L_0.$$

The social planner would maximize

$$\max(\lambda)^{\beta\alpha}(L_0-\lambda)^{1-\alpha},$$

and the solution is the same.

3.5.2. *Leontief's preferences.* Consider an exchange economy with two agents (Alice and Bob) and two goods (X and Y) and the Leontieff's preferences

$$u_i(x,y) = \min(a_i x, b_i y)$$

for some $a_i, b_i > 0$. Let the endowment be $\omega_i \in [0,1]^2$ for each agent i. We want to find Walrasian equilibria.

Case $p \gg 0$. Let's try first strictly positive prices. We compute that demand of agent i is going to be equal to

$$\mathbf{x}_A = \begin{bmatrix} x_i \\ y_i \end{bmatrix} = \left(\frac{p\cdot\omega_i}{p\cdot\gamma_i}\right)\gamma_i = \alpha_i\gamma_i,$$

$$\text{where}\gamma_i = \begin{bmatrix} b_i \\ a_i \end{bmatrix} \text{ and } \alpha_i = \frac{p\cdot\omega_i}{p\cdot\gamma_i}.$$

Because of part 3 of Lemma 2, we must have strict aggregate feasibility. Which means that

$$\mathbf{x}_A + \mathbf{x}_B = \omega_A + \omega_B = \omega.$$

Thus, we obtain a system of two equations

$$\alpha_A \gamma_A + \alpha_B \gamma_B = \omega, \text{ or}$$

$$\begin{cases} \alpha_A b_A + \alpha_B b_B &= \omega_X, \\ \alpha_A a_A + \alpha_B a_B &= \omega_Y, \end{cases}$$

with two unknowns α_A, α_B that, additionally, must satisfy the inequality constraints:

$$\alpha_A, \alpha_B \geq 0.$$

The form of the equilibrium depends on the existence and the number of solutions:

- If the above system has a unique solution, and the solution is neither $\alpha_A \gamma_A \geq \omega_A$ nor $\alpha_A \gamma_A \leq \omega_A$, then we can find prices $p \gg 0$ such that $p \cdot (\alpha_i \gamma_i - \omega_i) = 0$ for each i. (The reason is that if neither $\alpha_A \gamma_A \geq \omega_A$ nor $\alpha_A \gamma_A \leq \omega_A$, then the aggregate feasibility stated above implies that the vector $\alpha_i \gamma_i - \omega_i$ lies in either NW or SE orthants. Hence, it is orthogonal to a vector p from the SE orthant.) Notice that the two equations $p \cdot (\alpha_i \gamma_i - \omega_i) = 0$ for $i = A$ and B are not linearly independent; this is OK, because we already know that the prices are only determined bu to a multiplicative factor.
- If the above system has a unique solution, and the solution is such that $\alpha_i \gamma_i = \omega_i$, then any vector of prices constitutes an equilibrium.
- If the above system has a unique solution, and the solution is such that either $\alpha_A \gamma_A > \omega_A$ or $\alpha_A \gamma_A < \omega_A$, then there is no equilibrium with strictly positive prices.
- If the above system has infinitely many solutions, then each solution for which neither $\alpha_A \gamma_A \geq \omega_A$ nor $\alpha_A \gamma_A \leq \omega_A$ corresponds to an equilibrium. It is possible that there are infinitely many equilibria.
- If the above system has no solution (in which case, it must be that $\frac{b_A}{a_A} = \frac{b_B}{a_B}$), then there is no equilibrium with strictly positive prices.

Case $p_x = 0$. We can assume that $p_y = 1$. In such a case, the demand of agent i is a set

$$\left\{ \frac{\omega_{i,y}}{a_i} \gamma_i + \begin{bmatrix} \beta_i \\ 0 \end{bmatrix} : \beta_i \geq 0 \right\}.$$

The aggregate feasibility implies that

$$\sum_i \frac{b_i}{a_i}\omega_{i,y} + \beta_i \leq \sum_i \omega_{i,x},$$
$$\sum_i \omega_{i,y} \leq \omega_{i,y}.$$

The second inequality is satisfied trivially (with an equality). Thus, we can have an equilibrium of this form if $\sum_i \frac{b_i}{a_i}\omega_{i,y} \leq \sum_i \omega_{i,x}$.

Case $p_y = 0$. Reasoning as above, we show that an equiobrium with such prices exist if $\sum_i \frac{a_i}{b_i}\omega_{i,x} \leq \sum_i \omega_{i,y}$.

Exercise 6. Is it always true that the Leontief model has a Walrasian equilibrium?

4. Existence of equilibrium[16]

4.1. Basic intuition. Describe non-clearance when price of one good is zero.

Describe non-clearance when price of the other good is zero.

With strictly convex preferences, the demands change continuously.

With convex preferences, the demand correspondence is u.h.c.

4.1.1. *Example: Shoes.*

4.2. Fixed point theorems. All equilibrium proofs in economics rely on a fixed-point theorem. One of two most important "high" math results in economics. Two important types of fixed-point theorems are:

- a continuous function has a fixed-point,
- a correspondence has a fixed point.

4.2.1. *Function fixed point theorems.* Let K be a set and let $f : K \to K$ be a function from K to itself. We say that f has a fixed point, if there is a $k \sum K$ such that $f(k) = k$.

Theorem 1. *(Brouwer) Suppose that $K \subseteq \mathbb{R}^n$ is compact and convex. Every continuous function $f : K \to K$ has a fixed point.*

The convexity assumption is super important.

Exercise 7. Take $K = \{0, 1\}$, or $K = \left\{x \in \mathbb{R}^2 : \|x\| = 1\right\}$. Show that for each of these two sets, there are continuous functions that are onto, and that do not have the fixed point.

The assumption can be a bit relaxed: Instead of convexity, it is enough that K is obtained from a convex set by a continuous deformation.

The compactness is also important.

Exercise 8. Take $K = (0, 1)$, or $K = \mathbb{R}$. Show that for each of these two sets, there are continuous functions that are onto, and that do not have the fixed point.

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[16] $November 1, 2019$.

4.2.2. *Correspondence fixed point theorems.* The above result require the mapping from a set to itself to be a function. Sometimes, it is not enough: Unless we assume strict convexity, the demand is typically a correspondence, not a function.

A correspondence (i.e., a set-valued function) ϕ from the set X to the set Y is some rule that associates one or more points in Y with each point in X. Formally, it can be seen just as an ordinary function from X to the power set of Y, written as $\varphi : X \to 2^Y$. We write $\phi : X \rightrightarrows Y$.

We say that a correspondence $\phi : K \rightrightarrows K$ has a fixed point if there exists k such that $k \in \phi(k)$.

Suppose that $K \subseteq \mathbb{R}^n$. A Kakutani correspondence is a correspondence $\phi : K \rightrightarrows K$ from the set to itself such that

(1) $\phi(k)$ is nonempty for each $k \in K$,
(2) $\phi(k)$is convex and compact for each $k \in K$,
(3) ϕ is upper hemi-continuous: for each sequence of $k_n \to k$, and values $l_n \in \phi(k_n)$ such that $l_n \to l$, we have that $l \in \phi(k)$.

Theorem 2. *(Kakutani) Suppose that $K \subseteq \mathbb{R}^n$ is compact and convex. Every Kakutani correspondence $\phi : K \rightrightarrows K$ has a fixed point.*

4.2.3. *Generalizations.* Here are some generalizations of the above results. Mostly, to expand towards infinitely-dimensional spaces. The infinitely-dimensional generalization of Brouwer fixed point is the Shauder-Fixed Point Theorem:

Theorem 3. *(Shauder) Every continuous function from a convex compact subset K of a Banach space to K itself has a fixed point.*

Fan-Glicksberg. Other extensions: Please check out http://www.math.cmu.edu/∼omostovy/papers_

Convexity can be further generalized into objects that behave like convex. This leads to the fixed point theorems on Absolute Retracts or Absolute Neighborhood Retracts (ANRs).

4.3. **Some facts about correspondences.** Suppose that K, A, B are compact subsets of some Euclidean spaces.

4.3.1. *Product correspondence.* Let $\psi : K \rightrightarrows A$ and $\phi : K \rightrightarrows B$ be two correspondences. We can create a product correspondence $\psi \times \phi : K \rightrightarrows A \times B$ in the following way: for each $k \in K$,

$$\begin{aligned}(\psi \times \phi)(k) &:= \psi(k) \times \phi(k) \\ &= \{(a,b) : a \in \psi(k), b \in \phi(k)\}.\end{aligned}$$

Lemma 3. *1. If ϕ, ψ are convex-valued, then $\phi \times \psi$ is convex-valued.*

2. If ϕ, ψ are u.h.c., then $\phi \times \psi$ is u.h.c..

Exercise 9. Prove the Lemma.

4.3.2. *Optimal solution.* A special type of correspondences associate optimization problem with their optimal solutions. Let K be a compact set and let $f : K \times X \to \mathbb{R}$ be a continuous function. We consider a optimization problem

$$\max_{x \in X} f(x,k),$$

where we optimally choose x given a parameter of the problem $k \in K$. Because of the continuity and compactness, the value of the maximization problem

$$f_{\max}(k) = \sup_{x \in X} f(k,x)$$

is finite and it is attained by some x for each k. (The latter means that we can replace "sup" by "max".) Define the correspondence of the solution to the maximization problem

$$x_f^*(k) = \arg\max_{x \in X} f(k,x).$$

Lemma 4. *Suppose that K, X are compact subsets of Euclidean space and f is continuous.*

1. x_f^* is nonempty-valued and u.h.c.

2. If, additionally, f is quasi-concave in x for each k, then x_f^* is convex-valued.

Proof. We are going to show the first part. Suppose that $k_n \to k$,$x_n \in x_f^*(k_n)$ for each n, and $x_n \to x$. Then, $f_{\max}(k) \geq f(k,x) = \lim f(k_n, x_n) = \lim f_{\max}(k_n)$. If $f_{\max}(k) = \lim f_{\max}(k_n)$, then all inequalities can be replaced by equalities, which implies that $x \in x_f^*(k)$. Hence, the claim will be show if we can prove that the the value function $f_{\max}$ is lower semi-continuous. For this, take any $x' \in x_f^*(k)$ and notice that $f_{\max}(k) = f(k, x') = \lim f(k_n, x') \leq \lim f_{\max}(k_n, x')$.

We leave the second part as an exercise (you have already proven it in the Consumer Theory part of the class). □

A more complicated situation arises when the set of available choices changes with the parameter. From now on, assume that we have a correspondence $X_K : K \rightrightarrows X$ of available choices. Consider the problem,

$$\max_{x \in X_K(k)} f(x, k),$$

and define

$$f_{\max}(k) = \sup_{x \in X(k)} f(k, x),$$
$$x_f^*(k) = \arg\max_{x \in X(k)} f(k, x).$$

Lemma 5. *Suppose that K, X are compact subsets of Euclidean space and the correspondence $X_K : K \rightrightarrows X$ is Kakutani (i.e., u.h.c., convex and nonempty-valued,) and f is continuous.*

1.If, additionally, f is quasi-concave in x for each k, then x_f^* is convex-valued.

2. $f_{\max}(k)$ is upper semi continuous[17] *and x_f^* is nonempty-valued.*

3. If $f_{\max}(k)$ is lower semi-continuous, then x_f^* is upper hemi-continuous.

Proof. We verify the second claim. Suppose that $k_n \to K$, $x_n \in x_f^*(k_n)$. Because X is compact, we can assume (possibly by taking a subsequence) that x_n converges, $x_n \to x$. Because correspondence X_K is u.h.c., it must be that $x \in X_K(k)$. Hence, $f_{\max}(k) \geq f(x, k) = \lim_n f(x_n, k_n) = \lim_n f_{\max}(k_n)$. This shows upper semi-continuity of $f_{\max}$.

Suppose that $f_{\max}$ is also lower semi-continuous, hence continuous. Then, $f_{\max}(k) \geq f(x, k) = \lim_n f(x_n, k_n) = \lim_n f_{\max}(k_n) = f_{\max}(k)$ and all inequalities can be replaced by equalities. But it implies that $f_{\max}(k) = f(x, k)$, or that $x \in x_f^*(k)$. Hence, x_f^* is upper hemi-continuous. □

[17] Recall that a function f is lower semi-continuous, if for each $s^m \to s$, we have $\liminf_m f(s^m) \geq f(s)$ and upper semi-continuous if $\limsup_m f(s^m) \leq f(s)$

4.4. **Existence of equilibrium results.** We follow the discussion in Kreps (13), who, in turn, follows the classnotes of Vijay Krishna.

Theorem 4. *Suppose that the economy satisfies the following assumptions:*

(1) *For each consumer i, X_i is convex subset of $\mathbb{R}^L$, and the preferences are continuous and convex.*
(2) *For each firm j, Y_j is convex.*
(3) *Compactness: For each consumer i, X_i is compact and for each firm j.*
(4) *For each consumer i, the endowment ω_i belongs to the interior of X_i. For each firm j, its technology satisfies the possibility of inaction: $\mathbf{0} \in Y$.*

Then, the economy has a Walrasian equilibrium with non-negative prices.

The substantive assumptions are that the preferences are convex and continuous and technologies are convex.

Assumption 3 is important for the proof. Because compactness means that the technologies and consumption spaces are bounded (and closed), it is considered strong (Kreps calls it "very bounded economy"). There are many ways of relaxing it (see below). I personally don't care because in our world, everything is finite, hence compact. You can read more on relaxation of the assumption in Kreps.

Assumption 4 is used in a technical part of the argument. There are many other assumptions that deliver the same result.

4.5. **Proof of Theorem.** Let $P = \left\{p \in [0,1]^L : \sum_l p^l = 1\right\}$ be the simplex of normalized non-negative price vectors.

$$
\begin{aligned}
S_0 &= X_1 \times ... \times X_I \times Y_1 \times ... \times Y_j \times P \\
&= \{(x_1, ..., y_J, p) : \forall_i x_i \in X_i, \forall_j y_j \in Y_j, p \in P\}.
\end{aligned}
$$

be the space of individually feasible allocations and (bounded) prices.

(1) First, we are going to define correspondence
$$\phi : S_0 \rightrightarrows S_0.$$
(2) We show that ϕ is a Kakutani correspondence. We postpone one part of the proof that the correspondence is u.h.c. for later.

(3) We show that the fixed point of ϕ is a Walrasian equilibrium.
(4) We finish the proof that ϕ is u.h.c.

4.5.1. *Definition of correspondences.* In the first step of the proof, we are going to define correspondences

$$\phi_i^C : S_0 \rightrightarrows X_i \text{ for each } i,$$
$$\phi_j^P : S_0 \rightrightarrows Y_j \text{ for each } j,$$
$$\phi^0 : S_0 \rightrightarrows P,$$

and correspondence ϕ so that for each $s \in S_0$,

$$\begin{aligned}\phi(s) &= \phi_1^C(s) \times ... \times \phi_I^C(s) \times \phi_1^P(s) \times ... \times \phi_J^P(s) \times \phi^0(s) \\ &= \left\{(x_1, ..., x_I, y_1, ..., y_J, p) : \forall_i x_i \in \phi_i^C(s), \forall_j y_j \in \phi_j^F(s), p \in \phi^0(s)\right\}.\end{aligned}$$

The idea is that for each "old" allocation $s = (x_1, ..., y_J, p)$, the correspondence associates a set of "new" allocations:

- Consumers compute their wealth given "old" prices and market choices, and choose "new" optimal bundles: For each consumer i, define $w_i(s) = p \cdot \omega_i + \sum_j \theta_{ij} p \cdot y_j$ as a function of the "old" prices and production plans. Define

$$\phi_i^C(s) = x_i^*(p, w_i(s)).$$

- Firms chose "new" optimal plans given "old" prices: For each j, define

$$\phi_j^P(s) = y_i^*(p)$$

- Walrasian auctioneer chooses "new" prices to maximize the social value of the market imbalances: Let

$$\phi^0(s) = \arg\max_{p \in P} p \cdot \left(\sum_i x_i - \sum_i \omega_i - \sum_j y_j\right).$$

  The Walrasian auctioneer will increase the price of the goods that have surplus demand, and decrease the price of goods that are not demanded enough.

4.5.2. *Kakutani correspondence.* We are going to check that ϕ is a Kakutani correspondence

$$\phi : S_0 \rightrightarrows S_0$$

has the Kakutani property.

By Lemma 3, it is enough to check that each of the coordinate correspondences $\phi_i^C, \phi_j^P, \phi^0$ is Kakutani.

The fact that correspondences ϕ_j^P, ϕ^0 are Kakutani follows from Lemma 4.

The fact that correspondences $\phi_i^C$are nonempty- valued and convex valued, follows from Lemma 5.

The u.h.c. of ϕ_i^C will follow from Lemma 5 if we can show that the consumer's optimal utility is lower semi-continuous. We postpone the proof till step 4.

4.5.3. *Walrasian equilibrium.* Because S_0 is a compact and convex set, Theorem 2 applies and ϕ has a fixed point. We are going to show that the fixed point is a Walrasian equilibrium.

Suppose that $s = (x_1, ..., y_J, p)$ is a fixed point of the correspondence ϕ. Then,

- for each consumer i, $x_i \in x_i^* (p, w_i (s))$ and
- for each firm j, $y_j \in y_j^* (p)$.

Thus, to verify that allocation $(x_1, ..., y_J)$ with prices p is a Walrasian equilibrium, we need to check the aggregate feasibility. For this, notice that the budget constraint for each consumer implies that

$$p \cdot x_i \leq p \cdot \omega_i + \sum_j \theta_{ij} p \cdot y_j.$$

By summing it across all consumers, we get

$$p \cdot \sum_i x_i \leq p \cdot \omega + \sum_j p \cdot y_j.$$

Hence, the budget constraint implies that the payoff of the Walrasian auctioneer is never positive. On the other hand, if there is a good l such that

$$\sum_i x_i^l > \omega^l + \sum_j y_j^l,$$

then the auctioneer could have chosen $p^l = 1$ and $p^{l'} = 0$ for each $l' \neq l$ to get the positive payoff. Hence, the allocation satisfies the weak version of the aggregate feasibility.

4.5.4. *Demand correspondence is u.h.c.* We are still left to show that ϕ_i^C is u.h.c.. We use the fact that the initial endowment belong to the the interior of X_i. We start with a (very technical, but useful) Lemma.

Lemma 6. *Suppose that assumption 4 holds and X_i is convex.*
Then, for each $s = (x^0, p) \in S$ such that the firms maximize profits, each wealth level $w_i = w_i(s)$, each $x \in X_i$ such that $p \cdot x \leq w_i$, there is a sequence $x_m \in X_i$ such that $x_m \to x$ and $p \cdot x_m < w_i$.

Proof. Recall that

$$w_i(s) = w_i\left(x_1^0, ..., x_I^0, y_1^0, ..., y_J^0, p\right) = p \cdot \omega_i + \sum_j \theta_{ij} \left(\max_{y_j \in Y_j} p \cdot y_j^0\right).$$

Because the endowment is in the interior of the consumption space X_i, there exists $\varepsilon > 0$ and a bundle $\rho_i \in X_i$ such that $p \cdot \rho_i < p \cdot \omega_i - \varepsilon$. Because the technology contains the possibility of inaction, the profits of each firm are non-negative, $p \cdot y_j \geq 0$, which implies that

$$w_i - p \cdot \rho = p \cdot \omega_i + \sum_j \theta_{ij} \left(\max_{y_j \in Y_j} p \cdot y_j^0\right) - p \cdot \rho \geq p \cdot \omega_i - p \cdot \rho \geq \varepsilon > 0.$$

Take $x \in X_i$ such that $p \cdot x \leq w$. For each m, define $x_m = \frac{1}{m}\rho + \frac{m-1}{m}x$. Because X_i is convex, allocation x_m is individually feasible, $x_n \in X_i$. Moreover,

$$w_i - p \cdot x_m = \frac{m-1}{m}w_i - \frac{m-1}{m}p \cdot x + \frac{1}{m}w_i - \frac{1}{m}p \cdot \rho \geq \frac{1}{m}(w_i - p \cdot \rho) \geq \frac{1}{m}\varepsilon > 0.$$

□

Armed with the Lemma, we can proceed with the proof of u.h.c of teh consumer's correspondence.

Lemma 7. *Suppose that assumption 4 holds and X_i is convex. Then, the optimal utility*

$$u_i^*(s) = \max_{x \in X_i : p \cdot x \leq w_i(s)} u_i(s)$$

is lower semi-continuous.

Proof. Take any sequence $s^n = (x^n, p^n) \to s = (x, p)$. Let $x^* \in x_i^* (p, w_i (s))$ be the optimal allocation at s. By Lemma 6, there exists a sequence $x_m \to x^*$ such that $x_m \in X_i$ and for each n, $p \cdot x_m < w_i (s)$. Moreover, by the continuity of wealth and convergence of prices,

$$w_i (s^n) - p^n \cdot x_m \overset{m\to\infty}{\longrightarrow} w_i (s) - p \cdot x_m > 0.$$

Hence, for sufficiently high n, x_m satisfies the budget constraint at s^n. It follows that $u_i^* (s^n) \geq u_i (x_m)$ and

$$\liminf_n u_i (s^n) \geq u_i (x_m) \text{ for each } m.$$

As we take $m \to \infty$, $x_m \to x^*$, and the continuity of the utility function implies that

$$\liminf_n u_i (s^n) \geq u_i (x^*) = u_i^* (s).$$

Hence, u_i^* is lower semi-continuous. □

4.5.5. *Comments on the proof.* The proof is almost "game-theoretic": there are agents (consumers, firms and the auctioneer), with well-defined payoffs, that depend on their own actions, who choose their optimal best responses. In fact, when you see the proof of the existence of Nash equilibrium, you will realize that the latter is a simpler version of the proof of the existence of the Walrasian equilibrium.

The main difference is that here, we assume, that the set of actions that are available to some of the agents (i.e., consumers), depends on the actions taken by other players (i.e., prices, bu the auctioneer). In game theory - that is not good, as all actions are supposed to be taken simultaneously and independently.

4.6. **Other existence theorems.** If you don't like the compactness assumption of Theorem 4, there are other results that you may like better. For instance, the next result substantially weakens the compactness assumption.

Theorem 5. *Suppose that the economy satisfies the following assumptions:*

(1) *For each consumer i, X_i is convex subset of $\mathbb{R}^L$, and the preferences are continuous and convex.*

Additionally, preferences are semi-strictly convex: if $x \prec_i x'$*, then* $x \prec_i \alpha x' + (1-\alpha)\, x$ *for each* $\alpha \in (0,1)$*.*

(2) *For each firm* j*,* Y_j *is convex.*

(3) *Bounded relevant outputs: There exist a large number* $B < \infty$ *such that for any profile of outputs* $(y_j) \in \times Y_j$*, if* $\omega + \sum_j y_j \geq 0$*, then it must be that* $|y_j| \leq B$ *for each* j*.*

(4) *For each consumer* i*, the endowment* ω_i *belongs to the interior of* X_i*. For each firm* j*, its technology satisfies the possibility of inaction:* $\mathbf{0} \in Y$*.*

Then, the economy has a Walrasian equilibrium.

Comparing with Theorem 4, the assumption 1 about consumer preferences is strengthened a bit. On the other hand, the compactness assumption is significantly relaxed. Theorem 5 does not require the compactness of the consumption spaces. Instead, the last assumption essentially says that the relevant space of technologies is bounded, hence compact (it is almost like compactness).

The idea of the proof is to work with a bounded version of the economy, find the bounded equilibrium, and then release the bound and show that the existence of equilibrium is not affected.