9. Matching[27]

Matching models are special cases of a broader class of allocation problems, in which agents are allocated with "each other". Many of them are purely social, ie. they describe relations between groups of people, but they also include economic allocations, like workers to jobs, students to schools, tenants to dorm rooms, etc.

9.1. Marriage matching problem.

9.1. Marriage matching problem. Basic model of one-to-one matching with strict preferences was introduced in (Gale-Shapley 1962). This section is based on (Roth 1992) that contains all the classic results. The matching theory got a second life in and after 00s

- finite sets of men M and women W
  - Examples include men - women, workers - firms, students - schools, etc.
- Each man $m \in M$ has *strict* preferences over $W \cup \{m\}$, i.e., over women and staying single, $<_m$.
- Each woman w has *strict* preferences over $M \cup \{w\}$, i.e., over men and staying single, $<_w$.

Typically, preferences and the entire problem is purely ordinal. Also, typically, without loss of generality to assume that people have utility functions as long as the problem is finite. We denote by

- $u_m(w)$ the utility of man m from being matched with $w \in W \cup \{m\}$, and
- $v_w(m)$ the utility of woman w from being matched with $m \in M \cup \{w\}$.

If the problem is infinite, utility functions are more natural. In any case, because we will not consider randomized matchings here, there is a complete equivalence between strict preference orderings and utilities without indifferences.

If preferences are not strict, typically, we can break the indifferences. Everything that we say for the basic model applies also to the model in which preferences are broken.

[27] $December 5, 2019.$

Definition 18. *Matching* is one-to-one correspondence $\mu : M \cup W \to M \cup W$ that has order 2 (i.e., $\mu(\mu(x)) = x$) and such that if $\mu(m) \neq m$, then $\mu(m) \in W$, and similarly for w.

Matching is an allocation in this model. Some man and women can remain unmatched, otherwise, each man can marry only one woman, and vice verse. Let $\mathcal{M}(M, W)$ denote the set of matchings between men and women from sets M and W.

For each matching, let

$$U(m;\mu) = \begin{cases} u_m(\mu(m)) & \text{if } \mu(m) \in W, \\ u_m(0) & \text{otherwise} \end{cases}$$

be the utility of man m from matching μ. Similarly, define the utility $V(w;\mu)$ of woman w from matching μ.

We can think about matching problems as a special case of NTU-cooperative games (I, V), where $I = M \cup W$ and for each coalition $M' \subseteq M, W' \subseteq W$, we have

$$V(M' \cup W') = \{(u_m(\mu), u_w(\mu)) : \mu \in \mathcal{M}(M', W')\}.$$

9.2. **Stable matching.**

Definition 19. Matching μ is *stable* if it is

- *individually rational*: for each ind. x, x (weakly) prefers her match to being alone,
$$x \leq_x \mu(x),$$
- *pairwise stable*: no pair of men and women can block the matching: for each m, w,
$$\text{eitehr } w \leq_m \mu(m) \text{ or } m \leq_w \mu(w)$$
.

Cooperative concept. In fact, the above definition is identical to a (weaker) version of the core: mathcing is stable if it cannot be blocked by any 1- or 2 person coalition S, i.e., any S st. $|S| \leq 2$. It turns out that, because of the nature of the matching problem, the set of stable matching is equal to the core: Recall that matching is *blocked by coalition* $A \subseteq M \cup W$ if there exists (sub)matching $\mu' : A \to A$ such that $\mu'$is stable on A and everybody on A prefers μ' to μ. Matching μ is in the core if it is not blocked by any coalition.

Theorem 12. *The core is equal to the set of all stable matchings.*

Proof. If there is a deviating coalition, then clearly, there exists a blocking pair. But this means that the original matching wasn't stable. □

Using the utility notation, there is an equivalent definition of a stable matching:

Exercise 20. Show that matching μ is stable if and only if for each man $m \in M$,

$$U(m; \mu) = \max_{w \in \{w' \in W : v_{w'}(m) \geq V(w';\mu)\} \cup \{m\}} u_m(w),$$

and, for each woman $w \in W$,

$$V(w; \mu) = \max_{m \in \{m' \in W : u_{m'}(w) \geq U(m;\mu)\} \cup \{w\}} v_w(m).$$

For instance, the first equality says that the stable match utility of man m is equal to the best utility he can achieve among all available matches either by statying alone, or by matching with a woman w' who likes him more (not less) than her utility $V(w'; \mu)$.

Questions:

(1) Existence.
(2) Efficiency.
(3) Structure of the set of stable matchings.
(4) Comparative statics
(5) Incentives: strategy-proofness.
(6) Uniqueness.
(7) Characterization.

9.3. **Existence: Gale-Shapley algorithm.** Men-proposing algorithm (there is analogous women-proposing). Each man has a list of of women from top to bottom. Each woman has a notebook with all the offers she got.

The algorithm moves through rounds. At the beginning of round 0, each man start with the complete list of preferences, and each woman starts with her own name in the notebook.

At each round,

- each man approaches woman at the top of his list or does not approach anybody, if staying alone is on top of his list,
- each woman adds all her offers to the notebook and crosses out all the suitors except for the best offer, where, possibly, the best offer is to stay alone,
- each man that is rejected by his top woman crosses her out of her list.

At each round, the men's list grows shorter, the women's best current offer is getting better. Because men and women are finite, the algorithm must stop.

Formally, we define

- for each man m and woman w,
  - let $m_w^0 = \{w\}$ be the most attractive match for women w at the beginning of the process (i.e., being single),
  - let $w_m^0 = \max_{\leq_m} W \cup \{m\}$ be the most attractive match for man m at the beginning of the process.
- for each $n \geq 1$, each man m and woman w,
  - let $m_w^n = \max_{\leq_w} \{m_w^{n-1}\} \cup \{m : w_m^{n-1} = w\}$ be the most attractive match among the most attractive match from the previous round and all the men that rank w as the most attractive match in the previous round,
  - let $w_m^n = \begin{cases} w_m^{n-1} & \text{if } w_m^{n-1}{=}\emptyset \text{ or } m^n_{w_m^{n-1}} = m, \\ \max_{\leq_m} \{w : w <_m w_m^{n-1}, w \in W \cup \{m\}\} & \text{otherwise} \end{cases}$

    be the attractive match for man m in round n: either the same as the most attractive match in the previous round, or if the previous most attractive match is not available, the next best match.

Notice that

- for each woman w :$m_w^{n-1} \leq_w m_w^n$, or woman's utility weakly increases in each round,

- for each man m: $w_m^{n-1} \geq_m w_m^n$, or man's utility weakly decreases in each round,
- if , at some step, none of the inequalities is strict, then we have equalities in all subsequent steps. We say that the matching algorithm stops at the first n that it happens. We define $\mu(m) = w_m^\infty$ and $\mu(w) = m_w^\infty$. Notice that if$w_m^\infty \in W$, then man m must be most attractive man for woman w_w^∞ at the last round: $m_{w_m^\infty}^\infty = m$. A similar fact holds for women.

Because the number of agents is finite, the above properties mean that the algorithm has to stop at some point (we always go down the list and the lists and the number of lists are finite.)

Remark 5. The deferred acceptance algorithm is robust to men skipping their rounds and forgetting to approach their most attractive remaining match for one or more rounds. As long as in the end, all men approach their most attractive match, the algorithm will continue with exactly the same end result.
It is also robust to men getting in hurry and skipping approaching the women who are going to reject them.

Theorem 13. *The matching obtained in the men-proposing deferred acceptance is stable.*

Proof. First, notice that the outcome of the algorithm is a proper match: Either $\mu(m) = m$ or $\mu(\mu(m)) = \mu(w_m^\infty) = m_{w_m^\infty}^\infty = m$; similarly, either $\mu(w) = w$, or $\mu(\mu(w)) = w$.

Individual rationality is obvious (no man or no woman ever rejects herself if it is a best option left).

To check that there is no blocking pair, take any m, w that are not matched with each other.

- If man m likes woman w more than his match $\mu^*(m)$, it means that man m must have approached woman w during the process and clearly got rejected: there is n such that $w_m^{n-1} = w >_w w_m^n$.
- But it means that woman w preferred her temporary match in that round: $m_w^n >_w m$.

- Moreover, her current match is always (weakly) better than all previous, rejected ones: $\mu(w) = m_w^\infty \geq_w m_w^n >_w m$
- Hence, w prefers her current match. Thus, m and w are not a blocking pair.

□

Because everything is symmetric, we can define a women-proposing deferred acceptance to find another stable matching. Typically, there are many more matchings than those two.

9.4. **Efficiency.**

Definition 20. Stable matching $\mu$is M-optimal, if for each man m, for each other stable matching μ', $\mu(m) \geq_m \mu'(m)$.

Theorem 14. *The (men-proposing) deferred acceptance matching μ_M is M-optimal. It is also worst for women: for each other stable matching μ, for each woman w, $\mu_M(w) \leq_w \mu(w)$.The alternative, women deferred acceptance matching is W-optimal and worst for men.*

Proof. Suppose that μ is any other matching. Suppose that there is either w such that $\mu(w) < \mu_M(w)$ or there is m such that $\mu(m) >_m \mu_M(m)$. We are going to show that μ cannot be stable.

Because of the construction of the men-proposing algorithm, there must be the first round n such that either (a) there exists w such that $m_w^n >_w \mu(w)$, or (b) there exists man m such that $w_m^n <_m \mu(m)$.

Suppose that we have (a). In such a case, $m_w^{n-1} \leq_w \mu(w)$ and for all men m such that $w = w_m^{n-1}$, we have $w = w_m^{n-1} \geq_m \mu(m)$. Because $m_w^n >_w \mu(w)$, it means that there is man $m = m_w^n$, such that $w = w_m^{n-1}$ and $m >_w \mu(w)$. Because $\mu(w) \neq m$, it must be that $\mu(m) \neq w$. Hence, because $w_m^{n-1} = w \geq_m \mu(m)$ implies that $w >_m \mu(m)$. It follows that (m, w) is a blocking pair for matching μ.

If we don't have (a), then suppose that we have (b). Then, $w_{m'}^{n-1} \geq_{m'} \mu(m')$ for all men m' and $m_w^n \leq_w \mu(w)$ for all women w. Because $w_m^n <_m \mu(m)$, it must be that $w_m^{n-1} = \mu(m)$ and man m is rejected by woman $w = \mu(m)$ in the nth round of the algorithm. Let $m' = m_w^n$. Hence, $w = w_{m'}^{n-1} \geq_{m\prime} \mu(m\prime)$. Because $\mu(w) = m$, it must be that $\mu(m') \neq w$ and $w >_{m'} \mu(m')$. Hence, (m', w) is a blocking pair for matching μ'. □

Remark 6. Notice for future reference that the proof shows that if $\mu(m) > \mu_M(m)$, then

9.5. **Lattice structure of the set of stable matchings.** For any two stable matchings μ_0 and μ_1 define function λ:

- $\lambda(m) = \mu_0(m)$ if $\mu_0(m) >_m \mu_1(m)$ and $\lambda(m) = \mu_1(m)$ otherwise (thus, men is matched to his more preferred partner)
- $\lambda(w) = \mu_0(w)$ if $\mu_0(w) <_w \mu_1(w)$ and $\lambda(w) = \mu_1(w)$ otherwise (woman gets worse partner).

We also denote function λ as $\mu_0 \vee_M \mu_1$.

Theorem 15. *$\lambda = \mu_0 \vee_M \mu_1$ is a stable matching. All men prefer λ to either μ_0 or μ_1 and all women prefer μ_0 to λ (or, μ_1 to λ).*

Proof. Let

$$M_0 = \{m : \mu_0(m) >_m \mu_1(m)\},$$
$$M_1 = \{m : \mu_1(m) >_m \mu_0(m)\}$$
$$M' = \{m : \mu_0(m) = \mu_1(m)\}$$

Then, the above sets are disjoint decomposition of M. Also, notice that $\mu_i(m) \in W$ for each $m \in M_i$.

We show that λ is a good matching. On the contrary, suppose that $w = \lambda(m) = \lambda(m')$ for $m \neq m'$.

- both men m and m' prefer w to their matches under both matchings μ and μ'. Moreover each man is matched with w under one of the matchings.
- w.l.o.g. suppose that w prefers man m to man m' and suppose that w is matched with m' under matching μ'.
- because man m prefers w to $\mu'(m)$ and w prefers m to $\mu'(w) = m'$, (m, w) is a blocking pair and μ' is not stable. Contradiction.

Hence, it follows that $\lambda(M_0) \cap \lambda(M_1) = \emptyset$. Because μ_i is a bijection from M_i to $\mu_i(M_i) = \lambda(M_i)$, λ is a bijection from $M_0 \cup M_1$ to $\lambda(M_0 \cup M_1) = \lambda(M_0) \cup \lambda(M_1)$. Hence, μ is a well-defined matching.

Next, we show that λ is stable. Take any m and w and suppose that w prefers man m to $\lambda(w)$. Then, w prefers m to both $\mu_0(w)$ and $\mu_1(w)$. But this means that man m prefers $\mu_0(m)$ and $\mu_1(m)$ to w. In particular, m prefers $\lambda(m)$ to w. Hence, m and w is not a blocking pair.

□

Corollary 1. *The set of matchings form a lattice with the men optimal at the top and the woman optimal at the bottom. The lattice is distributive: (check in BOOK).*

9.6. **Rural hospital theorem.**

Theorem 16. *The set of people who stay single is the same in all stable matchings.*

Proof. Let $M^m(\mu)$ be the set of married men in matching μ. Then, for each stable matching μ

$$M^{\mathrm{m}}(\text{men optimal}) \supseteq M^{\mathrm{m}}(\mu) \supseteq M^{\mathrm{m}}(\text{women optimal}).$$

Otherwise, if man m would have higher utility in a non-optimal matching, which is impossible. A similar (reverse) observation holds for women. But for each match

$$|M^m(\mu)| = |W^m(\mu)|!$$

□

The name of the result due to A. Roth comes from the observation that in the resident-hospital matching, if some (typically, rural) hospitals are remain unmatched in some stable matching, it is not a problem of this particular stable matching, and it cannot be ameliorated by a different matching algorithm.

Remark 7. One of the benefits of the rural hospital theorem is that it helps in finding other stable matches, once we find one of them. For instance, suppose that you found woemn-optimal match. Then all men and women that are unmatched, will remain unmatched in any other, including men-optimal match. Hence, when running the men-proposing algorithm, we can safely ignore these men and/or women.

9.7. **Incentives.** In order to operate, the (men- or women proposing) deferred acceptance algorithm needs preferences as an input. But how do we now the preferences? We can of course ask. But, then, how do we now whether people (men and women) report truthfully? The problem is that the reports will determine outcomes. It is possible that somebody discovers that misreporting her preferences may lead to a more favorable outcome. This section examines this issue. We are going to define a matching mechanism as a mapping that chooses a stable matching on each matching market.

From now on, fix set of men M and women W. Let $\mathcal{M}(M,W)$ be the set of matchings on M and W. A matching market is defined as a tuple of strict preferences: $\left((\leq_m)_{m\in M}, (\leq_w)_{w\in W}\right)$. Let $\mathcal{A}(M,W)$ be the set of all matching markets.

A matching mechanism $\phi : \mathcal{A}(M,W) \to \mathcal{M}(M,W)$ assigns markets with a matching market A with matching $\phi(A)$. The mechanism is *stable* if for each A, matching $\phi(A)$ is stable in market A. An example of a stable matching mechanism is men-proposing deferred acceptance algorithm. Another example is the women proposing deferred acceptance.

We are going to consider the incentives to misreport preferences. Consider a market $A = \left((\leq_m)_{m\in M}, (\leq_w)_{w\in W}\right)$, and let $A_{x,\leq'_x}$ is a market where agent x reports that his or her true preferences are $\leq'_x$ instead of x and everybody except for x reports the same as in A.

Definition 21. Mechanism ϕ is *strategy-proof* if for each matching market $A \in \mathcal{A}$ for each x, each preference relation $\leq'_x$, agent x prefers outcome of the market under A than under his or her misreport $A_{x,\leq'}$:

$$(\phi(A))(x) \geq_x (\phi(A_{x,\leq'}))(x).$$

The strategy-proofness is a terminology from the cooperative literature. In the language of game theory, a mechanism is strategy proof if truth-telling is a Nash equilibrium of a game, where agents report preferences. A closely related notion in mechanism design is incentive compatibility.

Theorem 17. *There exists no strategy-proof and stable matching mechanism.*

Proof. It is enough to find a single market such that for each stable matching, at least one man or woman would like to restate his/her preferences. Suppose that there are two men and two women. Consider a market with preferences:

$$w_1 >_{m_1} w_2, w_2 >_{m_2} w_1,$$

$$m_2 >_{w_1} m_1, m_1 >_{w_1} m_2,$$

There are two stable matchings (a) (m_1, w_1) and (m_2, w_2) and (b) (m_1, w_2) and (m_2, w_1).

Suppose that the matching mechanism chooses (a). If woman w_2 misreports preferences m_1 (i.e., she prefers to stay single rather than marry m_2), then (b) is the only stable matching. But w_2 prefers m_1 to m_2, so she is better off misstating her preferences.

Similarly, if the mechanism picks (b), then man m_1 is better off reporting that he prefers $w_1 > \emptyset$ (i.e., that he prefers to be single than to marry w_2).

It is easy to use the above example to find problematic markets for a a general (not two-element) sets of men and women. □

Theorem 17 is quite disappointing. It is impossible to find a mechanism which always produces a stable matching, and that cannot be manipulated. Because the issue of incentives is so important for modern economic theory, the literature studied this issue very intensively. Overview of the results:

(1) Although fully strategic proof mechanism is impossible, it is possible to design mechanisms, in which the incentive problems only affect one side of the market. For instance, one can show that in the men-proposing deferred acceptance, truthtelling is always best response for men. So, all teh incenitve problems are on the women side.
(2) This observation is very helpful in situations in which, for whatever reasons, we know the preferences of one of the sides of the market. For example, it is possible that by law, the school prefer students according to some legally defined criteria (distance to the school, outcome on some exam, etc.). In such a case, the school preferences are known; the student-proposing algorithm becomes fully strategy-proof.

(3) Even if teh preferences are not fully known, it is possible to show that in some situations the incentive problem disappears. More precisely, one shows that if a woman receives the same match *under every stable matching* (or equivalently, under the men-optimal or women-optimal matchings), she has no incentives to misreport under the men-proposing algorithms. This should be intuitive, the incentives to misreport should only exist if one can gain. (One can actually show that if a woman gains through misreport under the men-proposing algorithm, then the resulting matching must be stable under original preferences.)
(4) It follows that if we know that the matching market has a unique stable matching. In such a case, none of the deferred acceptance algorithms has any incentive problem. Examples of such markets are given in the next section.
(5) Some of the recent papers examined approximate strategy-proofness and showed than in many markets, with some stochastic model of preferences, a serious incentive problem affects only a small group of agents. It happens in these papers that the stable matchings are very close to each other. The message of this literature, as far I understand it is that almost uniqueness is closely related to almost strategy-proofness. But, I have never seen a result that says this explicitly.

9.8. **Uniqueness.** Common preferences (a.k.a. 1-dimensional case)

Theorem 18. *Suppose that all men have the same preferences. Then, there is a unique stable matching.*

Proof. Because all men have the same preferences, we can arrange all women from the best to the worst. Consider matching, in which the best woman picks first, then the second woman picks avoiding the pick of the first, etc. It is easy to see that in the men-proposing or women-proposing algorithms select this matching. □

Special case (exactly 1-dimensional)

Example 20. Men and women are distributed on the interval $m, w \in [0, 1]$. For each man m, utilityof man m, $M(m, w)$, is increasing in w. For each w, utility and

$W(m, w)$ is increasing in m. Then, the stable matching is unique and it is possitively assortative.

Top-match property

Definition 22. Matching market has *top-match property* if for each subset of men $M' \subseteq M$ and for each subset of women $W' \subseteq W$, there exists man $\hat{m} \in M'$ and woman $\hat{w} \in W'$ such that both are most favorite matches in the subsets (man $\hat{m}$ is the most favorite men of woman $\hat{w}$ among all men in set M' and vice verse).

$$\hat{m} \in \arg\max_{m}$$

It is easy to see that top-match property (and strict preferences) implies uniqueness. **?** shows that this property is also necessary if we want to have unique matching in each subset.

10. Matching with transferable utility[28]

10.1. **Model and notation.** In this section, we assume that individuals have quasi-linear preferences over matches. Each man m and woman w has preferences represented by utility functions

$$M(m,w)+\theta \text{ for } w\in W\cup\{\emptyset\},$$
$$W(m,w)+\theta,\ \text{ for } m\in M\cup\{\emptyset\}.$$

Here, we can think about θ as a payment, or transfer. To simplify notation, we sometimes write $M(m,\emptyset)=M_0(m)$ and same for women. Also, for $m\in M\cup\{\emptyset\}$ and $w\in W\cup\{\emptyset\}$, let

$$f(m,w)=M(m,w)+W(m,w)$$

denote the joint production function. (We take $f(\emptyset,\emptyset)=-\infty$ although it is never going to play any role.)

- In the marriage model, the transfers correspond to bride-price, dowry, allocation of responsibilities in the household, division of household income, prenuptial agreement etc.
- More importantly, the model with transfers allow for matching workers and firms and wages!

A *matching with transfers* (μ,θ) is a pair of matching μ and a transfer function st. for each $x\in M\cup W$

$$\theta(x)+\theta(\mu(x))=0.$$

It follows that aggregate budget condition holds:

$$\sum_m \theta(m)=\sum \theta(w). \tag{10.1}$$

A matching with transfers (μ,θ) is stable if two conditions are satisfied:

- individual rationality: for each m and w, each c

$$M(m,m)\leq M(m,\mu(m))+\theta(m),$$
$$W(w,w)\leq W(w,\mu(w))+\theta(w),$$

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- pairwise stability: for each m and w, each ξ

$$\begin{aligned} \text{either } M(m,w)+\xi &\leq M(m,\mu(m))+\theta(m), \\ \text{or } W(m,w)-\xi &\leq W(w,\mu(w))+\theta(w). \end{aligned}$$

We have the following simple observation:

Lemma 10. *Matching wit transfers is pairwise stable if and only if for each m and w:*

$$M(m,w)+W(m,w)\leq M(m,\mu(m))+\theta(m)+W(w,\mu(w))+\theta(w). \tag{10.2}$$

Proof. Exercise. □

10.2. **Characterization of stable matching.** Define the aggregate welfare of a matching μ :

$$\begin{aligned} \mathcal{W}(\mu,\theta) = & \sum_{m:\mu(m)=\emptyset} M_0(m,\mu(m)) + \sum_{w:\mu(w)=\emptyset} W_0(w) \\ & + \sum_{m:\mu(m)\neq\emptyset} M(m,\mu(m))+\theta(m)+W(m,\mu(m))+\theta(m) \\ = & \sum_m M(m,\mu(m)) + \sum_w W(w,\mu(w)) \\ = & \mathcal{W}(\mu), \end{aligned}$$

i.e., it does not depend on the transfers.

Theorem 19. *For any matching μ, the following two are equivalent:*

1. *There exist transfers θ such that (μ,θ) is a stable matching with transfers,*
   (a) $\mu \in \arg\max_{\upsilon:\upsilon \text{ is a matching}} W(\upsilon)$

The Theorem presents a characterization of a stable matching with transfers: matching (with some transfers) is stable iff it maximizes welfare

10.2.1. *Proof of direction: If a matching is stable stable, then maximizes welfare.* Let (μ,θ) be a stable matching with transfers. Let υ be any other matching. We are going to show that $\mathcal{W}(\mu)\geq W(\upsilon)$.

For each m st. $\upsilon(m) = \emptyset$, teh individual rationality of μ implies that

$$M(m, \emptyset) \leq M(m, \mu(m)) + \theta(m).$$

For each w st. $\upsilon(w) = \emptyset$, the individual rationality of μ implies that

$$W(w, \emptyset) \leq W(w, \mu(w)) + \theta(w).$$

For each m, st. $\upsilon(m) \neq m$, let ξ_m be a transfer that would make m indifferent between match $(\mu(m), \theta(m))$ and $(\upsilon(m), \xi_m)$:

$$M(m, \upsilon(m)) + \xi(m) = M(m, \mu(m)) + \theta(m) \tag{10.3}$$

Then, the pairwise stability condition applied to potential blocking pair $(m, \upsilon(m), \xi_m)$ implies that

$$W(m, \upsilon(m)) - \xi(m) \leq W(\upsilon(m), \mu(\upsilon(m))) + \theta(\upsilon(m)). \tag{10.4}$$

(Note that if m receives teh transfer, his match $\upsilon(m)$ must pay up the same amount.) Adding up (10.3) and (10.4), we obtain

$$\begin{aligned} & M(m, \upsilon(m)) + W(m, \upsilon(m)) \\ \leq & M(m, \mu(m)) + W(\upsilon(m), \mu(\upsilon(m))) - \theta(m) + \theta(\upsilon(m)). \end{aligned}$$

(This observation follows also from lemma 10.

Notice that the welfare of matching υ can be written as the sum of welfare of single men, single women, and matched pairs,

$$\begin{aligned} \mathcal{W}(\upsilon) = & \sum_m M(m, \mu(m)) + \sum_w W(w, \mu(w)) \\ = & \sum_{m:\upsilon(m)=m} M(m, \emptyset) + \sum_{w:\upsilon(w)=w} W(w, \emptyset) \\ & + \sum_{m:\upsilon(m)\neq m} M(m, \upsilon(m)) + \sum_w W(m, \upsilon(m)). \end{aligned}$$

By the above inequalities, this is not larger than

$$\leq \sum_{m:\upsilon(m)=m} M(m,\mu(m)) + \theta(m) + \sum_{w:\upsilon(w)=w} W(w,\mu(w)) + \theta(w)$$
$$+ \sum_{m:\upsilon(m)\neq m} \left(M(m,\mu(m)) + W(\upsilon(m),\mu(\upsilon(m)))\right) + \theta(m) + \theta(\upsilon(m))$$
$$= \sum_m M(m,\mu(m)) + \sum_w W(w,\mu(w)) + \sum_m \theta(m) + \sum_w \theta(w)$$
$$= \mathcal{W}(\mu)$$

by the aggregate budget condition (10.1).

10.2.2. *Proof of direction: If matching maximizes welfare, then stable.* Solve linear programming problem:

$$W^* = \max \sum_{m,w} \mu(m,w)\left(M(m,w) + W(m,w)\right)$$
$$+ \sum_{m,w} \mu(m,\emptyset)\left(M(m,w) + W(m,w)\right) + \sum_{m,w} \mu(\emptyset,w) W(\emptyset,w)$$
$$\text{st.} \forall_{m,w} \mu(m,w) \geq 0,$$
$$\forall_m \sum_w \mu(m,w) + \mu(m,\emptyset) = 1,$$
$$\forall_w \sum_m \mu(m,w) + \mu(\emptyset,w) = 1.$$

Here, $\mu \in \Delta((M \cup \{\emptyset\}) \times (W \cup \{\emptyset\}))$. Because the solution set is spanned by the matchings (Birkhoff–von Neumann Theorem), a solution is a proper matching, and it is an efficient matching.

To find the transfers, we consider a dual problem:

$$J^* = \min \sum_m U(m) + \sum_w V(w)$$
$$\text{st.} \forall_{m,w} U(m) + V(w) \geq M(m,w) + W(m,w),$$
$$\forall_m U(m) \geq M(m,\emptyset),$$
$$\forall_w V(w) \geq W(w,\emptyset).$$

By the duality theory (see below), the values of the primal and the dual problems are equal

$$J^* = W^*.$$

Suppose that pair $(U, V)$is a solution to the dual problem, and μ is a solution to the primal problem. Then,

$$\begin{aligned} U(m) + V(\mu(m)) &\geq M(m, \mu(m)) + W(m, \mu(m)) \text{ if } \mu(m) \neq \emptyset, \\ U(m) &\geq M(m, \mu(m)) \text{ if } \mu(m) = \emptyset, \\ V(w) &\geq W(w, \mu(m)) \text{ if } \mu(w) = \emptyset, \end{aligned}$$

which implies that

$$\begin{aligned} J^* =& \sum_m U(m) + \sum_w V(w) \\ \geq& \sum_{m:\mu(m)=\emptyset} M(m, \emptyset) + \sum_{w:\mu(w)=\emptyset} W(w, \emptyset) \\ &+ \sum_{m:\mu(m)\neq\emptyset} M(m, \mu(m)) + W(m, \mu(m)) \\ =& W(\mu) \geq J^*(\mu) \end{aligned}$$

and the inequality turns into equality only if all the above constraints are equal!

Let

$$\begin{aligned} \theta(m) &= 0 \text{ if } \mu(m) = m, \\ \theta(w) &= 0 \text{ if } \mu(w) = w, \\ \theta(m) &= M(m, \mu(m)) - U(m) \text{ for all other } m. \end{aligned}$$

Easy to check that the matching μ with transfers θ is stable! Indeed, IR for men is satisfied because their utility is equal to $U(m) \geq M_0(m)$ by constraints. Same for women.Moreover, suppose that m, w is a blocking pair with a transfer θ: Then

$$\begin{aligned} M(m, w) - \theta &> U(m), \\ W(m, w) + \theta &> V(w). \end{aligned}$$

But the sum of the above inequalities contradicts the constraint of the dual problem!

10.2.3. *Duality theory.* Let $b \in \mathbb{R}^m, c \in \mathbb{R}^n$ and let A be $m \times n$ matrix. Consider two finitely dimensional linear programming problems.

- Primal problem:$\max c^T x$ st. $Ax \leq b$ and $x \geq 0$, where $x \in \mathbb{R}^n$
- Dual problem: $\min b^T y$ st. $A^T y \geq c$ and $y \geq 0$, where $y \in \mathbb{R}^m$.

Theorem 20. *(Strong Duality Theorem). The values of the primal and the dual problems are the same.*

To apply to the matching,

- assume that $f(m, w) \geq 0$ (add constant to normalize), and take

$$c = \begin{bmatrix} f(m_1, w_1) \\ \ldots \\ f\left(m_1, w_{|W|}\right) \\ f(m_1, \emptyset) \\ f(m_2, w_1) \\ \ldots \\ \ldots \\ f\left(m_{|M|}, \emptyset\right), \\ f(\emptyset, w_1), \\ \ldots \\ f\left(\emptyset, w_{|W|}\right) \end{bmatrix}, b = \begin{bmatrix} 1 \\ \ldots \\ \ldots \\ \ldots \\ \ldots \\ 1 \end{bmatrix}_{n_M + n_W},$$

  and matrix A that is $(|M| + |W|) \times ((|M| + 1) \times (|W| + 1) - 1)$-dimensional.

10.3. **One-dimensional case.** Suppose that men and women are distributed on the interval $m, w \in [0, 1]$. Define the joint marriage production function $f(m, w) = M(m, w) + W(m, w)$.

Definition 23. Say that f is strictly supermodular, if for any $m < m'$ and $w < w'$

$$f(m, w) + f(m', w') > f(m, w') + f(m', w).$$

Theorem 21. *If f is strictly supermodular, then, the unique stagble matching with transfers is assortative. If $-f$ is supermodular, then the unque stable matching with transfers is negatively assortative.*

Proof. Suppose that two matched pairs in a matching are not matched assortatively. Then, a rematch increases the welfare from thetwo pairs, and, as a consequence, the total welfare. Apply Theorem . □

Compare with NTU case!

- TU: supermodularity (i.e., cross derivative of the joint production function)
- NTU: first derivative of the utilities.

10.4. **Matching with contracts.** The models of matching with or without transfers have a nice common generalization. The idea is that, apart from two matching partners, each match can be formed under some conditions that are chosen by the partners. For instance, a man and a woman can form a marriage through a regular law-recognized form of a wedding, they can cohabit without any formal ties, or they can sign some sort of civil agreement, or anything else. The form of the marriage presumably affects the utility of he spouses.

More importantly, when a worker matches with a firm, they sign a contract hat specifies work conditions, hours, whether the work is done from office or at home, benefits, work uniform, etc. All of these match conditions are describe in a contract, the details of which presumably affect the parties utility from the match.

Suppose that C is the set of all possible contractual conditions. A match of man m with woman w under contract c gives him and her the utility

$$M(m, w, c) \text{ and } W(m, w, c).$$

To make the notation consistent, we describe the utility of agent x staying alone as a function of a conract (or form of staying alone):

$$M(m, m, c) \text{ and } W(w, w, c);$$

a typical assumption could be that c does not affect the above utilities.

A matching with contracts is a tuple (μ, c) of a matching μ and a mapping $c : M \cup W \to C$ such that for each x,

$$c(x) = c(\mu(x)).$$

This condition ensures that each match has a well-defined contract.

A matching with contracts (μ, c) is stable if two conditions are satisfied:

- individual rationality: for each m and w, each c

$$M(m, m, c) \leq M(m, \mu(m), c(m)),$$
$$W(w, w, c) \leq W(w, \mu(w), c(w)),$$

- pairwise stability: for each m and w, each c

$$\text{either } M(m, w, c) \leq M(m, \mu(m), c(m)),$$
$$\text{or } W(m, w, c) \leq W(w, \mu(w), c(w)).$$

of staying alone. in principle deach worker w from working at firm f under contractual conditions c are given by $u_w(f, c)$. The utility of firm f from worker w under contractual conditions c are $v_f(w, c)$. The utility of staying alone is respectively, $u_w(\emptyset)$ and $v_f(\emptyset)$.

We may consider the following [29] version of the Gale-Shapley algortithm to find a stable matching.

- Initialization:
  - Each woman has a list of potential options: man-contract pairs. Its current tentative allocation is to stay alone.
  - Each man has a list of available options: woman-contract-pairs.
- In each round, each man m has some (w, c) at the top of his list. He approaches woman w and offers her contract c, i.e., offers her option (m, c).
  - Each woman collects the offers and chooses the best offer. If the best offer is better than her current tentative allocation, then the current tentative allocation becomes the current best offer. The woman crosses out all offers that are worse than then her current (possibly, newly updated) tentative allocation. All the crossed out offers are rejected.
  - All men whose offers are rejected, cross out the offer from their lists.
  - The process continues until some round is passed without any crossing out.

[29] Men-proposing. Of course, there is an analoguous firm-proposing version.

As the original deferred acceptance algorithm, the algorithm stops in finite time. One can show that the outcome of the algorithm (the current tentative match for the firm, and the top position on the list for the worker) is a stable match.

Exercise 21. 1. Show that the stable match obtained in the previous point is worker-optimal.

2. Show that an appropriate version of the rural hospital theorem applies in this setting.