2. Pareto-optimal allocations[13]

2.1. **Definition.** Suppose that we have economy where each agent has preferences over feasible allocations (that are derived from their own preferences over consumption spaces as in Section 1.3.4).

Definition 2. A feasible allocation x is *Pareto-optimal* if for any other feasible allocation y, if $y \succ_i x$ for some agent, then there is an agent $j \neq i$ such that $y \prec_j x$.

if a feasible allocation x is Pareto-optimal, then any other feasible allocation that makes somebody better-off must make somebody else strictly worse-off.

Let $P \subseteq X$ be the set of Pareto-optimal allocations.

2.2. **Finding the utility possibility frontier.** Define the utility-possibility set

$$UP = \{(u_1(x_1), ..., u_I(x_I)) : x \in X\}.$$

The Pareto-optimal allocations correspond to the utilities from the upper-boundary of this set. (Some examples). If everything works well, the set is sufficiently regular and we can find the Pareto-optimal allocations as the solution to the following program:

Proposition 1. *Suppose that the commodity sets are standard (i.e., $X_l = \mathbb{R}$ or $\mathbb{R}_+$ for each commodity l), each agent i's preferences are strongly monotone and continuous, and $X_i = Z$ for each agent i. Then, an allocation $x^* = (x_1^*, ..., y_J^*)$ is Pareto-optimal if and only if, for some i,*

$$x^* \in \arg\max_{x \in X} u_i(x) \text{ st. } u_j(x) \geq u_j^* \text{ for each } j \neq i,$$

where $u_j^* = u_j(x^*)$. Moreover, at the optimum, the constraint holds with equality.

Proof. One direction, namely if x^* is Pareto-optimal, then it is a solution to the maximization problem, is trivial.

The proof of the other direction is a bit more complicated. Suppose that $x^*$is a solution to the problem that is not Pareto-optimal. Hence, there must be an allocation

$$x' = (x_1', ..., x_I', y_1', ...y_J')$$

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and player $j \neq i$ such that $u_j(x') > u_j(x^*)$ and $u_k(x') \geq u_k(x^*)$ for each $k \neq j$. We will show that there exists a feasible allocation x'' such that $u_i(x'') > u_i(x^*)$ and $u_k(x_0) \geq u_k(x^*)$ for each $j \neq i$.

The idea is that allocations $x''$and x' differ only with respect to the consumption plans of agents j and i: we are going to reallocate some fraction of the consumption bundle from j and give it to i.

Because the commodity sets are standard, there is a consumption bundle $b < x_j$ such that $b \in Z = X_j$. (Consumption bundle b is bad). For very small $\varepsilon > 0$, construct

$$x''_j = (1-\varepsilon)\, x'_j + \varepsilon b = x'_j - \varepsilon \left(x'_j - b\right).$$

Because preferences of agent j are continuous, and because $u_j(x'_i) > u_j(x^*)$, for sufficiently small $\varepsilon > 0$, we are going to have $u_j(x''_i) \geq u_j(x^*)$. Construct

$$x''_i = x'_i + \varepsilon \left(x'_j - b\right).$$

Then, $x''_i \in X_i$. (This is because the assumption on the consumption space of agent i as well as the standard commodity sets.) Moreover, by strict monotonicity, $u_i(x'') > u_i(x') \geq u_i(x^*)$.

Construct x'' from x' by replacing the allocations of players j and i by x''_j and x''_i. Clearly x'' is feasible. (Convince yourself why.) Thus, we find an allocation x'' that is feasible and that makes each consumer at least as well-off as than under x^*, and consumer i strictly better off. That is not good as it contradicts the fact that $x^*$is the solution to the maximization problem.

The last claim follows from the proof. □

2.3. **Examples.**

2.3.1. *Linear preferences.*

Example 9. Consider exchange economy with two goods, X and Y, total endowment $\omega = (1,1)$ and two agents Alice and Bob with preferences $u_i(x,y) = a_i x + b_i y$, where we assume that $a_i, b_i > 0$ for each i. Describe the utility-possibility frontier. Find the set of all Pareto-optimal allocations.

We want to solve

$$\max_{x_A+y_A} a_A x_A + b_A y_A \text{ st. } a_B\left(1-x_A\right)+b_B\left(1-y_A\right)=u_B.$$

The constraint implies that

$$y_A = 1-\frac{u_B}{b_B}+\frac{a_B}{b_B}-\frac{a_B}{b_B}x_A.$$

Because $y_B \in [0,1]$, we have $x_A \in \left[1-\frac{u_B}{a_B}, 1-\frac{u_B}{a_B}+\frac{b_B}{a_B}\right]$. We can substitute to the original problem, to get

$$\max_{x_A}\left(\frac{a_A}{b_A}-\frac{a_B}{b_B}\right)x_A+1-\frac{u_B}{b_B}+\frac{a_B}{b_B} \text{ st. } \max\left(1-\frac{u_B}{a_B},0\right)\leq x_A \leq 1+\frac{1}{a_B}\min\left(b_B-u_B,0\right).$$

The solution depends on the sign of the coefficient with x_A. If $\frac{a_A}{b_A}=\frac{a_B}{b_B}$, then all non-wasteful allocations are Pareto-optimal.

Assume $\frac{a_A}{b_A}>\frac{a_B}{b_B}$ (the remaining case is solved analogously). In other words, Alice cares relatively more about the good X rather than Y. If $b_B > u_B$, then $x_A=1$,and $y_A = 1-\frac{1}{b_B}u_B$. Given Bob's utility, Alice's utility is equal to

$$a_A + b_A - \frac{b_A}{b_B}u_B.$$

If $b_B \leq u_B$, then $y_A=0$, and $x_A = 1-\frac{u_B-b_B}{a_B}$. Alice's utility becomes equal to

$$a_A+\frac{a_A}{a_B}b_B-\frac{a_A}{a_B}u_B.$$

The utility -possibility frontier's Alices allocations are

$$\left\{(x_A,0): x_A\in[0,1]\right\}\cup\left\{(1,y_A): y_A\in[0,1]\right\}.$$

In other words, we first start giving good X to Alice, and the rest ot Bob; once we do not have any good X left, Alice strats taking the other good.

2.4. **Assignment problem.**

Example 10. There are I agents and L objects. Each agent i needs at most one object, receives utility u_{il} from getting object l and utility u_{i0} if she receives nothing. A (pure, possibly wasteful) allocation is a mapping $a: I \to L\cup\{0\}$ such that for each i, i', l if $a(i)=a(i')\in L$, then $i=i'$. (We allow for wasteful allocations because maybe some agents do not like certain objects.)

If you need a concrete example to focus attention, think about students as agents and rooms in the dorm as objects. Other applications are offices for professors, students to universities.

We are going to describe the Pareto-optimal allocations in the above example. To make things simpler, we assume that each agent has a strict preference ranking over her options, $u_{i,l} \neq u_{i,l'}$ for any i and $l, l' \in L \cup \{0\}$ such that $l \neq l'$.

Consider the following way of choosing allocations. First, assign a priority order: bijection $\pi : I \to \{1, ..., |I|\}$. Then, each agent in order $\pi(1), \pi(2), \pi(3), ...$ makes a pick of his favorite object among all remaining (yet unpicked) objects. . The objects chosen by the agents are removed from the set of remaining objects. We refer to such an algorithm as the *Serial Dictatorship*.

More formally, let $L^0 = \{\emptyset\}$and for each $k = 1, 2, 3, ...$,define inductively

$$l^k = \arg \max_{l \in \left(L \setminus L^{k-1}\right) \cup \{\emptyset\}} u_{\pi^{-1}(k), l},$$

$$L^k = L^{k-1} \cup \left\{l^k\right\}.$$

In other words, l^k is an object chosen at stage k by agent $\pi^{-1}(k)$. Such agent chooses freely from objects that are not chosen by agents with a higher priority.

We are going to show than any allocation obtained by the Serial Dictatorship is Pareto-optimal. Let a be an allocation that is obtained from the Serial Dictatorship with priority order π . Suppose that a' is another allocation $a' \neq a$. We show that a' is not Pareto-optimal. Because $a' \neq a$, the set $\{i : a(i) \neq a'(i)\}$ is not empty. Let

$$i^* = \arg \min_{i : a(i) \neq a'(i)} \pi(i)$$

be the highest priority individual whose allocations differ under a and a'. By the construction, it must be that $a'(i) \notin L^{k-1} \setminus \{\emptyset\}$ (otherwise, we would vi9olate feasibility with some higher priority individuals). But then, again by the construction, $u_{i,a'(i)} < u_{i,a(i)}$.

Exercise 4. Show that any (pure, wasteful) Pareto-optimal allocation can be obtained from the Serial Dictatorship under some priority assignment.

Notice that the total number of all priority orderings is equal to $I!$ and potentially very large. Although not all priority orderings lead to different allocations (convince yourself so), they might (for instance, when all agents have exactly identical preferences, each priority ordering leads to a different allocation.) Thus, there is potentially a very large number of Pareto-optimal allocations.

2.5. **Quasi-linear economy.** We can define the welfare of the allocation as the sum of the utilities: For each $x \in X$, let

$$W(x) := \sum_i u_i(x_i).$$

This definition is typically not useful, because the utility levels of different agents do not typically have any meaning, as they are defined up to monotone transformations.

The definition becomes useful if all consumers have quasi-linear preferences over the same numeraire good, say good L. Then, as we know, the numeraire allows us to directly compare the (non-linear) utilities from other allocations. We have the following powerful result:

Proposition 2. *If all consumer preferences are represented by quasi-linear utilities* $u_i(x_i) = u_i^0(x_{i,-l}) + x_{i,l}$ *over good* L*, then*

$$W(x) = \sum_i u_i^0(x_{i,-l}) + \omega_L,$$

and a feasible allocation x *is Pareto-optimal if and only if it maximizes welfare among all feasible allocations:*

$$x \in \arg \max_{x' \in X: \sum x_{i,L} = \omega_L} \sum_i u_i^0(x_{i,-l}).$$

Proof. We show first that if x is Pareto-optimal, then it must maximize welfare. Suppose that x, x' are feasible allocations such that is an allocation such that $W(x) < W(x')$. We are going to create a feasible allocation x'' that Pareto-dominates x.

Indeed, fore ach j,

$$\begin{aligned}
x''_{i,-L} &= x'_{i,-L}, \\
y''_j &= y'_j, \text{ and} \\
x''_{i,L} &= x'_{i,L} + u_i\left(x_i\right) - u_i\left(x'_i\right) + \frac{1}{I}\left[W\left(x'\right) - W\left(x\right)\right].
\end{aligned}$$

The individual utility from the new allocation is equal to

$$\begin{aligned}
u_i\left(x''_i\right) &= u_i\left(x'_i\right) + u_i\left(x_i\right) - u_i\left(x'_i\right) + \frac{1}{I}\left[W\left(x'\right) - W\left(x\right)\right] \\
&= u_i\left(x_i\right) + \frac{1}{I}\left[W\left(x'\right) - W\left(x\right)\right] > u_i\left(x_i\right).
\end{aligned}$$

To check that the allocation is feasible, we only need to look at the aggregate feasibility of the numeraire (because nothing else changed). But,

$$\begin{aligned}
\sum_i x''_{i,L} &= \sum_i x'_{i,L} + \sum_i \left(u_i\left(x_i\right) - u_i\left(x'_i\right)\right) + I\frac{1}{I}\left[W\left(x'\right) - W\left(x\right)\right] \\
&= \sum_i x'_{i,L} + W\left(x\right) - W\left(x'\right) + \left[W\left(x'\right) - W\left(x\right)\right] \\
&= \sum_i x'_{i,L}.
\end{aligned}$$

Hence, x'' is feasible and x'' Pareto-dominates x.

For the other direction, suppose that x maximizes welfare among all feasible allocations that satisfy the strong aggregate feasibility for the numeraire. Consider any other allocation x'. We have

$$W\left(x\right) \geq W\left(x'\right).$$

Hence, it cannot be that all consumers are weakly better off and at least one of the is strictly better off under x'. □

2.5.1. *Assignment problem with quasi-linear preferences.* Consider an assignment problem like in Example 10, but where there is an additional good "money" and the agents have quasi-linear utility over money (and the previously described utility from objects). The existence of money dramatcially changes the set of Pareto-optimal allocations. In fact, the above result shows that they can be found as solutions to the maximization problem:

$$\max_{a:I\to L\cup\{0\}} \sum_i u_{i,a(i)}$$
$$\text{st.if } a\left(i\right) = a\left(i'\right) \in L, \text{then } i = i'.$$

The constraints make sure that the allocation is feasible.

The above problem is actually easier to solve when we expand the space of allocations by allowing lotteries in the sense of Example 7. Let $\alpha_i \in \Delta\left(L \cup \{0\}\right)$ be a random allocation of consumer i. Let $\alpha = (\alpha_i)$ be a (not necessarily feasible) random allocation. Consider the following problem:

$$\max_{\alpha=(\alpha_i)} \sum_i \sum_{l\in L\cup\{0\}} \alpha_{il} u_{il}$$
$$\text{st.} \sum_i \alpha_{il} \leq 1 \text{ for each } l.$$

The constraint implies that each object l cannot be allocated with a probability above 1 (it is possible that nobody wants some good; if these are preferences.)

This is a linear programming problem! We understand such problems very well, we have Kuhn-Tucker conditions to solve them, or simplex method. More importantly, such a problem has discrete (i.e., not random) solutions. Conivince yourself that any such non-random solution is a solution to the original discrete problem, hence it is a Pareto-optimal allocation.