6. Multiple markets[21]

One of the assumptions of the Walrasian model from Section 3 is that there is a single market. This assumption shows up in two places

- the statement of the consumer's (and firm's) problem: each consumer trades on one market, with a single budget balance,
- there is one aggregate feasibility constraint for each good.

This assumption is not appropriate for many situations. Typically, different goods are traded on different markets. We buy food on supermarket. We buy and sell shares and other financial assets on stock market. We save and borrow (buy and sell future consumption) in a bank. Etc. Is the Walrasian assumption that all transactions are done on one market a realistic simplification? A model of multiple markets will allow us to clarify that.

Another important example where consideration of multiple market is necessary is uncertainty. We discuss the equilibrium with uncertainty in the next section.

6.1. Model of multiple markets.

For simplicity, we restrict ourselves to pure exchange economy.

We assume that there are S markets. Each market s is characterized by the subset of goods $L_s \subseteq L$ that can be traded on the market. For each good l, we define the set of markets on which l is traded.

$$S_l = \{s : l \in S\}.$$

We are going to assume that $\bigcup_s L_s = L$, which implies that $S_l \neq \emptyset$ for each l, or that each good is traded somewhere. However, it is possible that some goods are traded on multiple markets, or, possibly, everywhere.

Each market will allow the consumers to trade their own goods (endowments) for new bundles of goods. We have the following changes with the basic single-market model:

[21] $November 21, 2019$.

- Each consumer i needs to decide what share of the endowment $\omega_{is} \in \mathbb{R}^{L_s}$ to bring to the market s. For each good, we will have endowment constraint

$$\sum_{s \in S_l} \omega_{i,s,l} = \omega_{i,l},$$

or, that the total is endowment of good l used in all markets where the good is traded is equal to the total endowment of good l.
- For each market s we have a vector of prices $p_s \in \mathbb{R}^{L_s}$. If the same good is traded on two different markets, prices may differ.
- Each consumer i is going to choose an allocation $x_{is} \in \mathbb{R}^{L_s}$ of goods traded on market s. The choice has to satisfy i's budget constraint on this market:

$$p_s \cdot x_{is} \leq p_s \cdot \omega_{is}.$$

- The total consumption of consumer i is given by a vector $x_i \in \mathbb{R}^L$ such that

$$x_{il} = \sum_{s \in S_l} x_{isl} \text{ for each } l.$$

Definition 6. A *(Radner) equilibrium* in the exchange economy with multiple markets is an allocation $(x_{isl}, \omega_{isl})_{i,s, \text{ and } l \in L_s}$ and a collection of price vectors for each market (p_{sl}) and such that

(1) Consumer maximization: each consumer i's choice $(x_{isl})_{s,l}$ is a solution to the problem

$$\max_{(x_{isl})_{s,l}, (\omega_{isl})_{s,l}} u_i(x_{i1}, ..., x_{iL}) \text{ such that}$$

$$\text{st.}$$

$$\text{buduget constraint:} \sum_{l \in L_s} p_{sl} x_{isl} \leq \sum_{l \in L_s} p_{sl} \omega_{isl} \text{ for each } s,$$

$$\text{endowment constraint:} \sum_{s \in S_l} \omega_{i,s,l} = \omega_{il} \text{ for each } l,$$

$$\text{total consumption:} x_{il} = \sum_{s \in S_l} x_{isl} \text{ for each } l.$$

(2) Feasibility:
   (a) The allocation is individually feasible for each agent i: $x_i \in X_i$,

(b) The allocation satisfies (weak) market-by-market aggregate feasibility:

$$\sum_i x_{ils} \leq \sum_i \omega_{ils}, \text{ for each } l.$$

Notice that a Walrasian equilibrium (Definition 3) is a special case of the Radner equilibrium, in which there is only one market and all goods are traded on the same market. Comparing to the Walrasian Equuilibrium , there are two differences.

- Each allocation has to satisfy budget constraints market-by-market.
- Each allocation must be feasible in each market.

Goods that are traded on multiple markets can be freely transferred between them through the choice of the endowment on each market and the fact that the total consumption is a simple sum of goods traded on each market.

6.2. **Examples.**

Exercise 12. There are two markets: Food Market, on which Apples and Bananas are traded and Electronics Market, on which one buys and sells Computers and other Devices. There are four agents, all of them with utility

$$u_i(a_i, b_i, c_i, d_i) = a_i b_i c_i d_i.$$

The endowments of the agents are

$$\omega_1 = (w, 1, 1, 1),$$
$$\omega_2 = (1, w, 1, 1),$$
$$\omega_3 = (1, 1, w, 1),$$
$$\omega_4 = (1, 1, 1, w),$$

for some $w \geq 0$. Find the Radner equilibrium. Show that the Radner allocation is not Pareto-optimal (ine the sense of the defintion from Section 2).

Solution 1. Notice that the utility is equivalent to $\log u_i = \log a_i + \log b_i + \log c_i + \log d_i$ (taking a logarithm is a monotonic transformation). The preferences are strongly monotonic, which implies that all prices are strictly positive, the Walras Law holds, and the budget constraints are satisfied with equality. Each consumer is solving the optimization problem

$$\max_{a_i,b_i,c_i,d_i} \log a_i + \log b_i + \log c_i + \log d_i$$
$$\text{st.} p_a a_i + p_b b_i = p_a \omega_{ia} + p_b \omega_{bi},$$
$$p_c c_i + p_d d_i = p_c \omega_{ic} + p_d \omega_{di}.$$

Because the problem is strictly conmvex, we can find solutions by looking at the FOCs of the Lagragian. Let λ_i^F and λ_i^E be the multipliers associated with, respectively, the Food and the Electronics markets. Then, we have FOCs:

$$a_i: \ \frac{1}{a_i} = \lambda_i^F p_a$$
$$b_i: \ \frac{1}{b_i} = \lambda_i^F p_b$$
$$c_i: \ \frac{1}{c_i} = \lambda_i^E p_c$$
$$d_i: \ \frac{1}{d_i} = \lambda_i^E p_d.$$

The budget constraints and the above equations imply that It follows that

$$p_a a_i = p_b b_i = \frac{1}{2}\left(p_a \omega_{ia} + p_b \omega_{bi}\right),$$

and an analoguous equation for the other market. In other words,

$$a_1 = \frac{wp_a + p_b}{2p_a} = \frac{1}{2}w + \frac{p_b}{2p_a}, a_2 = \frac{1}{2} + w\frac{p_b}{2p_a}, a_3 = a_4 = \frac{1}{2} + \frac{p_b}{2p_a},$$

(because apples are only traded on the Food Market, we can ignore the subscript that refers to the food market). The market clearing conditions are

$$\frac{1}{2}w + \frac{p_b}{2p_a} + \frac{1}{2} + w\frac{p_b}{2p_a} + 1 + \frac{p_b}{p_a} = w + 3,$$

which implies that

$$\frac{p_b}{p_a} = 1.$$

We have analogous equations for other goods $p_a = p_b$ and $p_c = p_d$. Hence, the Radner equilibrium allocation is

$$a = \left(\frac{1+w}{2}, \frac{1+w}{2}, 1, 1\right),$$
$$b = \left(\frac{1+w}{2}, \frac{1+w}{2}, 1, 1\right),$$
$$c = \left(1, 1, \frac{1+w}{2}, \frac{1+w}{2}\right),$$
$$d = \left(1, 1, \frac{1+w}{2}, \frac{1+w}{2}\right).$$

Clearly, such an allocation is not Pareto-optimal, because equal allocation

$$l_i = \frac{w+3}{4} \text{ for each } i \text{ and } l$$

would be an improvement.

6.3. **Pareto-optimality.** As we have seen in the above example, the Radner equilibrium with multiple markets is typically not going to be Pareto-optimal. The reason is that the the market-by-market feasibility is more restrictive than the aggregate feasibility. Thus, fewer allocations are allowed under RE.

It turns out that adding an extra good that can be traded in all markets may restore Pareto-efficiency. More precisely, we consider a model of the economy described in subsection 6.1, but with an additional good M that (a) can be traded across all markets, (b) the endowment for each i is equal to $\omega_{iM} = 0$ and the endowment can be split into positive and or negative quantities, but (c) the consumption of m_i on each market is restricted to be 0, $X_i \subseteq \{(x_1, ..., x_L, x_M) : x_M = 0\}$ (nobody can leave a market with a non-zero possession of money).

We are going to show that a Rander equilbrium of the multiple market economy with the extra good M that is traded in all markets is equivalent to the Walrasian equilibrium of the economy in which all goods are traded on the same market (and without good M, which is useless in this case). Formally, let

- E_R be the set of all Radner equilibria $(x_{isl}, \omega_{isl}, \omega_{iMl}, p_{sl}, p_{sM})$ in the economy with good M,

- E_W be the set of all Walrasian equilibria (x_{il}, q_l) in the economy described in subsection 1.3, i.e. without money and with only one single market.

Theorem 10. *(a) For each* $(x_{isl}, \omega_{isl}, \omega_{iMl}, p_{sl}, p_{sM}) \in E_R$ *such that all* $p_{sM} > 0$ *for each* s *and the Walras Law holds, define*

$$x_{il} = \sum_{s \in S_l} x_{isl}.$$

Then, there are prices $q \in \mathbb{R}^L$ *such that* $(x_{il}, q_l) \in E_W$*.*

(b) For each $(x_{il}, q_l) \in E_W$*, there is* $(x_{isl}, m_{is}, \omega_{isl}, \omega_{iMl}, p_{sl}, p_{sM}) \in E_R$ *such that* $p_{sM} > 0$ *and for each consumer* i *and good* l*,*

$$x_{il} = \sum_{s \in S_l} x_{isl}.$$

Together with the First Welfare Theorem, Theorem implies that each Radner equilibrium in the economy with money, is Pareto-efficient. Glorious money!

Proof. Part (a). Notice first that because the price of money is positive everywhere, we can assume w.l.o.g. that it is equal to $p_{sM} = 1$ for each s (we can always normalize prices on each market). Second, notice that if good l is traded on different markets $s \neq s'$, than its price on each market can be assumed equal. To see it, suppose that $p_{ls} > p_{ls'}$. If there is anybody with strictly positive consumption on market s, then such a agent is not optimizing: it would be better to sell good l on market s for money (i.e., for reduced endowment exposure), use the money (i.e., increased endowment)to purchase the same amount on market s' and use the savings to purchase perhaps more goods anywhere. Because of the Walras Law, it would improve such agent's utility. If there is no positive consumption on market s, the Walras Law implies that nobody wants to buy this good, and we can assume that price is lowered to $p_{ls'}$ without affecting equilibrium.

To summarize, we can find prices q_l such that $q_l = p_{ls}$ for each $s \in S_l$ and each l. Next, we check that allocation $x = (x_{il})$ with prices q is a Walrasian equilibrium. To do so, we need to show that each consumer is optimizing. On the contrary, suppose that for some i, there is an allocation $y_i \in X_i$ that satisfies the budget constraint

$$\sum_l y_{il} q_l \leq \sum \omega_{il} q_l$$

and such that $u_i(y_i) > u_i(x_i)$. We are going to use y to construct an allocation z in the Radner world that is feasible under the orrginal Radner equilibrium and that it has the same utility as y. This is going to lead to a contradiction with the fact that x was an optimal choice in the Radner world.

For each good l, fix market $s_l \in S_l$. Construct an allocation $z_i = (z_{isl}, \omega_{isl})$ such that for each l,

$$z_{is_l l} = x_{il}, \omega_{is_l l} = \omega_{il},$$
$$z_{isl} = 0, \omega_{isl} = 0 \text{ for each } s \neq s_l.$$

Let

$$\omega_{isM} = \sum_{l \in L_s} p_{sl} z_{isl} - \sum_{l \in L_s} p_{sl} \omega_{isl} = \sum_{l \in L_s} q_l z_{isl} - \sum_{l \in L_s} q_l \omega_{isl}. \tag{6.1}$$

The endowment choice is individually feasible:

$$\sum_s \omega_{isM} = \sum_s \left(\sum_{l \in L_s} p_{sl} z_{isl} - \sum_{l \in L_s} p_{sl} \omega_{isl} \right) = \sum_l q_l x_{il} - \sum_l q_l \omega_{il} = q \cdot x_i - q \cdot \omega_i = 0.$$

(WE used the Walras Law in the above equalities.) By construction, allocation z_i has the same total consumption as allocation y_i: hence, $u_i(\sum z_{is}) = u_i(y_i)$. The claim follows.

Part (b). We can proceed backwards - define prices $p_{sM} = 1$ and $p_{sl} = q_l$ for each s and l. We define the endowment choice as in (6.1). Finally, we show that the allocation is an optimal choice in the Radner world. The details are left as an exercise. (See also the proof of Proposition 3 below.) □

7. Uncertainty[22]

An important application of the multiple market model is to model uncertainty. For simplicity, we restrict the discussion to exchange economies.

[22] $November 21, 2019$.

7.1. **Uncertainty.** We consider economy with I consumers, and L_0 "goods". The endowments, preferences, (and technologies if we added production) can be affected by the realization of yet unknown state of the world $s \in S$, where S is finite. The consumption is done only AFTER the state is realized. However, the agents may also plan ahead, and possibly, reach an agreement on the future consumption BEFORE they learn the state.

No matter what they do, the allocations of goods and teh consumption bundles will typically depend on the state. To capture this, we use the trick intorduced in Section 1.4.5. We expand the notiion of a good to "state-contingent goods" that, from now on, we also call simply goods. A good x_{ls} is interpreted as "good" l consumed in state s. There are $L = L_0 \times S$ of such goods. Each consumption bundle is a

$$x_{11}, x_{12}, ..., x_{1S}, ..., x_{L1}, ..., x_{LS}.$$

The utility, in principle, may depend on the entire consumption bundle. Typically, we think about the utility from the *ex ante* point of view, before the markets are realized. Depending on the attitude over the uncertainty, the utility can have the expected utility form, or any other utility from uncertain allocations.

A feasible allocation has to satisfy the aggregate feasibility. Because each good is state-contingent, it means that the resource equation (1.1) has to be satisfied in each state for each good separately.

We consider different versions of the economy:

- Radner with only "spot markets". Suppose that the agents don't do anything before the state is realized. They wait until the state is realized and then, they start trading goods. There will be different markets for each state; we refer to such markets as "spot markets". The only goods traded on market s are state-contingent goods for this state s. This is the simplest model of dealing with uncertainty. Agents can only trade once the state and endowments
- Arrow-Debreu equilibrium: There is one market, presumably before the state is realized (period 0), in which the agents can trade all state-contingent goods. There are no other markets. Thus, the Arrow-Debreu model is essentially the same as the Walrasian equilibrium.

- Radner with "spot markets" and extra finanical asset(s) traded in period 0. There are spot markets, and there is a market in period 0. In period 0, one can only trade artificial (in the sense that thyey are not included in the original L_0 goods) financial (in the sense they do not enter utility) goods called "*assets*". Each asset is defined as a vector of promises, one for each state, to deliver a given amount of period 1 "finanical" good in this state. The period 1 finiancial good can be traded on the spot market in the state.
- Radner with Arrow-Debreu securities: It is Rdner with a very particular class of assets called There are S "spot" markets, where he agent trade goods in each state separately. Before that, and anticipating theprices on the spot markets, the agent trade certain type of financial asset called *Arrow-Debreu securities*.
- Radner with complete or incomplete markets:

As we reark above, in all the versions of the economy, the state-by-state god-by-good aggregate feasibility must be satisifed. The exact form of what "budget-balance" mean will depend on the version of the equilibrium.

Preferences:

- We assume that the consumers have some preferences over uncertainty, i.e., over state-contingent consumption bundles (or Savage acts in the language of the decision theory). The definitions do not make any restrictions on the form of the utility. However, in application, we typically assume that the preferences have the form of expected utility:

$$u_i\left(x_{11}, ..., x_{LS}\right) = \sum_s \pi_{is} u^*_{is}\left(x_{1s}, ..., x_{Ls}\right),$$

  where u^*_{is} is *the state-dependent utility function,* and π_{is} are the "subjective" probabilites.
  - If $u^*_{is} = u^*_i$ does not depend on the state, we refer to it as *the state-independent von Neuman-Morgenstern utility* function.
  - If $\pi_{is} = \pi_s$ do not depend on individual, we refer to them as *common probabilities.* (They do not need to be "objective" in whatever sense).

- There are preferences that are equivalent to expected utility. For example: $u_i(x_{11}, ..., x_{LS}) = \prod_{ls} x_{ls}^{\alpha_{ls}}$ where $\alpha_{ls} > 0$.

- The main advantage of the expected utility (and equivalent forms) is that it is separable across states. Hence, we can consider consumer optimization problems in each state. A very important property of such preferences (in fact a defining property of the expected utility in some sense) is time-consistency: optimal allocation chosen in period 0 remains optimal in each of the state (if the budget constraints do not change, of course).
- Non-expected utilities are possible - the definitions are the same. But, there are problems. See below.

7.2. **Radner with spot markets.** Probably the most natural way to trade under uncertainty is to assume that the agents have no special tools in dealing with it. The Radner equilibrium with spot markets given by Definition 6 is essentially a Walrasian Equilibrium state by state. We repeat the definition for the sake of completeness:

Definition 7. A *(Radner) equilibrium with spot markets only* is an allocation $(x_{isl})_{i,s,l}$ of state contingent goods, and a collection of asset price vectors $(p_{sl})_l$ for each spot market s such that

(1) Consumer maximization: each consumer i's choice $(x_{isl})_{s,l}$ is a solution to the problem

$$\max_{z_{is},(x_{isl})_{s,l}} u_i(x_{i1}, ..., x_{iSL}) \tag{7.1}$$
$$\text{st.} \sum_{l \in L} p_{sl} x_{isl} \leq \sum_{l \in L} p_{sl} \omega_{isl} \text{ for each } s,$$

The constraints are the budget constraints for each market.

(2) Feasibility:
   (a) The allocation is individually feasible for each agent i: $x_i \in X_i$,
   (b) The allocation satisfies (weak) aggregate feasibility in state-contingent physical goods:

$$\sum_i x_{isl} \leq \sum_i \omega_{isl} \text{ for each } s, l.$$

.

7.3. **Arrow-Debreu model.** At another extreme is an economy, where you only trade once, before the state is realized.

An Arrow-Debreu equilibrium is a (Radner) equilibrium, in which all agents trade in state-contingent goods on one market, opened in period 0. The right way to think about it is that the market opens at date 0, before the uncertainty is resolved. At this date, the agents trade in promises of delivering goods, rather than goods itself, because the goods are not realized yet.

Definition 8. A *(Arrow-Debreu) equilibrium* in the exchange economy with multiple markets is an allocation $(x_{isl})_{i,s,l}$ and a collection of price vectors for each market $(p_{sl})_{s,l}$ and such that

(1) Consumer maximization: each consumer i's choice $(x_{isl})_{s,l}$ is a solution to the problem

$$\max_{(x_{isl})_{s,l}} u_i(x_{i1}, ..., x_{iSL}) \tag{7.2}$$
$$\text{st.} \sum_s \sum_l p_{sl} x_{isl} \leq \sum_s \sum_l p_{sl} \omega_{isl},$$

The constraint (a) says that each agent's position must balance on market-by-market. The constraint (b) says that the agent consumes the total purchased amount on each market.

(2) Feasibility:
   (a) The allocation is individually feasible for each agent i: $x_i \in X_i$,
   (b) The allocation satisfies (weak) aggregate feasibility in state-contingent physical goods:

$$\sum_i x_{isl} \leq \sum_i \omega_{isl} \text{ for each } s, l.$$

The difference between Definitions 7 and 8 is that there is only one budget-balance condition here (one market). More precisely, the constraint in problem (7.2) is strictly weaker: it can be obtained by adding up constraints in problem (7.1) across states.

It follows from the FWT that the Arrow-Debreu allocation is Pareto-optimal, as long as the Walras Law holds. Hence, there is no further room for improvement(at least, not if we are not willing to abandon state-by-state aggregate feasibility. That can be done with some external insurance schemes that allow to transform goods from one state into another). [23]

7.4. **Uncertainty: Radner with spot markets and financial assets.** The Arrow-Debreu allocation is Pareto-optimal, but its implementation has two problems:

- it requires trading a large number of state-contingent goods,
- it requires lots of commitment in the sense that all trade occurs in delivery promises.

It turns out that the above problems can be dealt with under a more Radner like market structure, but with new artificial financial goods called *assets*. An asset is defined as a vector $r \in \mathbb{R}^S$ and we interpret it as a promise to deliver $r_s \in \mathbb{R}$ "dollars" (i.e., special "financial" good) in state s. We refer to r_s as the state s return of asset r. The " financial" means that it is made up, and that its value does not come from its consumption value, or as an input in technology, but from the equilibrium.

The assets have no consumption value. Their main role is that allow the agents to trade their wealth across states.

We assume that there are two types of markets:

- A single market in period 0: there is a trade in all assets $\mathcal{R}$.
  - The assets are traded in period 0 market (i.e., pre-market). The position in asset r (i.e., the allocation of asset r) by consumer i is denoted as $z_{ir} \in \mathbb{R}$.
  - The assets can be traded both in positive and negative quantities. The latter corresponds to borrowing, or short sales.
  - The initial endowment of each asset is 0. The budget constraint in period 0 together with the initial 0 endowment in assets says that
    $$\sum_{r \in \mathcal{R}} q_r z_{ir} \leq 0,$$
    that one cannot spend more than 0 on the assets. Here, q_r is the price of asset r in the period 0 market.

[23] We need to be careful about the notion of Pareto-dominance used here. Because in the Arrow-Debreu world we are looking from the point of vew of period 0, it is an ex ante notion. This is not necessarily the same as Pareto-optimality in each state.

– We assume *strict* feasibility in period 0 market:

$$\sum_i z_{ir} = 0. \text{ for each } r$$

We do it, because (at least without any further assumptions) we do not restrict the asset to be "goods" (it is possible that $r_s < 0$ for some or all states).

- S spot markets (oppen in period 1), where additionally to s-contingent goods, the agents receive their investment returns r_s.
  – The assets have no consumption value.
  – Their main role is that they affect the budget constraint in the spot markets s

$$\sum_{l \in L} p_{sl} x_{isl} \leq \sum_{l \in L} p_{sl} \omega_{isl} + \tilde{q}_s \sum_{r \in \mathcal{R}} z_{ir} r_s \text{ for each } s.$$

Here, $\tilde{q}_s$ is the price of the value of money, or more precisely, the value of one unit of the investment return[24]. It converts one unit of return of investment into one accounting unit of a budget constraint in spot market s. The last term of the right hand-side has an interpretation as the value of the entire investment return in state s.

  – The sum of the returns in state s across all agents is equal to

$$\sum_i \sum_{r \in \mathcal{R}} z_{ir} r_s = \sum_{r \in \mathcal{R}} \left(\sum_i z_{ir} \right) r_s = 0,$$

where the inequality comes from the fact that $\sum_i z_{ir} = 0$ and $r_s \geq 0$. Thus, the returns on the assets redistribute "wealth" among the agents in each period. Zero-sum investment - there is no production of value.

We have the following definition of an equilibrium.

[24]In the class, we used the following analogy: supposed that the accounting unit on the spot market is "dollar" and all the investments are measured in "euros", then $\tilde{q}_s$ is the exchange rate. If all the value of money The real value of not fixing the price of money to 1 is that all for the possibility that money is not valued and we are back to the original Radner equilibrium in Definition 7. (See also Remark 3 below.)

Definition 9. A *(Radner) equilibrium with financial assets* is a tuple $(x, p, z, q, \tilde{q})$ of an allocation $x = (x_{isl})_{i,s,l}$ of state contingent goods, $z = (z_{ir})_{i,r}$ of assets $r \in \mathcal{R}$, and a collection of asset price vectors $q = (q_r)_{r \in \mathcal{R}}$ for period 0 market, and $(p, \tilde{q}) = ((p_{sl})_l, \tilde{q}_s)_s$ for the spot markets such that

(1) Consumer maximization: each consumer i's choice $(x_{isl})_{s,l}$ is a solution to the problem

$$\max_{z_{is}, (x_{isl})_{s,l}} u_i(x_{i1}, ..., x_{iSL}) \tag{7.3}$$
$$\text{st.} \sum_{r \in \mathcal{R}} q_r z_{ir} \leq 0$$
$$\sum_{l \in L} p_{sl} x_{isl} \leq \sum_{l \in L} p_{sl} \omega_{isl} + \tilde{q}_s \sum_{r \in \mathcal{R}} z_{ir} r_s \text{ for each } s,$$

The constraints are the budget constraints for each market.

(2) Feasibility:
   (a) The allocation is individually feasible for each agent i: $x_i \in X_i$,
   (b) The allocation satisfies (weak) aggregate feasibility in state-contingent physical goods:

$$\sum_i x_{isl} \leq \sum_i \omega_{isl} \text{ for each } s, l.$$

   (c) The allocation satisfies strict aggregate feasibility in financial goods:

$$\sum_i z_{ir} = 0 \text{ for each } r \in \mathcal{R}.$$

In other words, each player i buys or (short) sells assets in day 0 and then uses them to relax or tighten the budget constraint in the spot markets. We assume that the prices of the assets in the market 0 and the spot markets are the same; we could also assume that they are different (as MWG does), but it would not change anything).

Some examples of financial assets:

- simple saving plan: asset $r = (1, ..., 1)$ is a promise to deliver 1 "dollar" in each state (i.e., regardless of the state).
- *Arrow-Debreu security* for state s_0 is a promise to deliver 1 dollar only in state s_0: $r^{s_0} = \begin{cases} 1, & s = s_0 \\ 0, & \text{otherwise.} \end{cases}$ There is one Arrow-Debreu security for each state.

- A state s_1 to state s_2 trade: asset r such that $r = \begin{cases} -1, & s = s_1 \\ 1, & s = s_2 \\ 0, & \text{otherwise.} \end{cases}$

Remark 3. Because the financial assets do not enter consumption, we do not have any natural assumption like monotonicity and their prices can be negative or positive. The same applies to the spot prices of money $\tilde{q}_s$. For any Radner equilibrium $(x, p, z, q, \tilde{q})$, we have Radner equilibria $(x, p, -z, -q, -\tilde{q})$. This should be clear: if we replace the allocations and prices by their negatives, than each budget constraint stays and holds exactly unchanged.

An interpretation of this fact is that the finanical assets, including money has no "natural" interpretation in the Radner model. Everything is "make-believe" and its value depends on our interpretation of it. For instance, $\tilde{q}_s$ is the spot market price of the value of"investment". If we interpret the notion of "investment" as borrowing (which we do when we replace demands z by $-z$), the conversion from the unit of "return" to the accounting unit changes as well.

Question : can we show that, say, spot market prices $\tilde{q}_s$ are always either all positive or all negative? Or is it possible in a Radner equilibrium to have prices $\tilde{q}_s$ positive for some states or negative for others?

7.5. **Radner with Arrow-Debreu securities.** Here, we assume that all the Arrow-Debreu securities are available for trade: Let $\mathcal{R}_{AD} = \{r^s : s \in S\}$ be the set of Arrow-Debreu securities.

It turns out that the Radner equilibrium with Arrow-Debreu securities is equivalent to the Arrow-Debreu equilibrium.

Proposition 3. *Suppose that $x \in \prod_i X_i$ is a feasible (in the Arow-Debreu sense, i.e., the sense of Section 7.3) consumption plan that satisfies strict aggregate feasibility, i.e., $\sum_i x_{isl} = \sum_i \omega_{isl}$ for each s, l. The following statements are equivalent:*

(1) *(x, p^*) is an equilibrium in the Arrow-Debreu model for some vector of spot prices $p^* \in \mathbb{R}^{LS}$.*

(2) $(x, p, z, q, \tilde{q})$ *is a Radner equilibrium with Arrow-Debreu securities for some some vector of spot prices* $p \in \mathbb{R}^{LS}$ *and* $\tilde{q}_s = 1$ *such that* $\tilde{q}_s > 0$ *for each* s*, asset demands* $z \in \mathbb{R}^{I\mathcal{R}_{AD}}$ *and asset demand prices* $q \in \mathbb{R}^{\mathcal{R}_{AD}}$*.*

Proof. Direction 1) $\to$ 2). We are going to choose prices $q \in \mathbb{R}^{\mathcal{R}_{AD}} = \mathbb{R}^S$ and allocations $z \in \mathbb{R}^{I\mathcal{R}_{AD}} = \mathbb{R}^{IS}$ of the Arrow-Debreu securities and show that, together with ADE allocations and prices, they form a Radner equilibrium. There are four steps:

Step 1: Prices and allocations. Let

$$\begin{aligned} p_{sl} &= p^*_{sl} \\ q_r &= 1 \text{ for each } r \in \mathcal{R}_{AD}, \\ \tilde{q}_s &= 1, \\ z_{ir^s} &= p_s \cdot x_{is} - p_s \cdot \omega_{is}. \end{aligned} \tag{7.4}$$

In other words, the position in asset r^s is equal to the difference between the value of consumer i's ADE purchase in state s and her endowment in this state, where the value is measured according to the spott market prices p_s.

Step 2. Budget constraints in Radner. Notice first that such demands for the assets satisfy period 0 budget constraint:

$$\sum_{r \in \mathcal{R}_{AD}} q_r z_{ir} = \sum_{s \in S} (p_s \cdot x_{is} - p_s \cdot \omega_{is}) = \sum_{s \in S} p_s \cdot x_{is} - \sum_{s \in S} p_s \cdot \omega_{is} \leq 0,$$

where the last inequality is satisfy due to the budget constraint in ADE. Moreover, notice that if consumer i chooses such demands for assets, then allocation x_{is} is feasible (in the sense that if satisfies budegt constraints) in the spot market s.

Step 3. Optimality in Radner. Finally, we can check that, given prices, such demands (for assets and goods) are optimal in the Radner world. The idea is that if there were other demands for physical goods and assets x'_i, z'_i that satisfy all Radner equilibrium budget constraints (i.e., constraints of problem (7.3)), then

$$\begin{aligned} \sum_{s,l} p_{s,l} x'_{isl} &\leq \sum_{s,l} p_{sl} \omega_{isl} + \sum_{s} \sum_{r \in \mathcal{R}_{AD}} z'_{ir} r_s \\ &= \sum_{s,l} p_{sl} \omega_{isl} + \sum_{r \in \mathcal{R}_{AD}} z'_{ir} \leq \sum_{s,l} p_{sl} \omega_{isl}, \end{aligned}$$

where the last equality comes from the fact that $q_r = 1$ and the asset budget constraint in period 0 implies that $\sum_{r\in\mathcal{R}_{AD}} z'_{ir} = \sum_{r\in\mathcal{R}_{AD}} q_r z'_{ir} \leq 0$. Hence, allocation x'_i is available in the ADE (problem (7.2)). Because the consumer optimizes with x_i, it must be that $u_i(x'_i) \leq u_i(x_i)$. It follows that x_i is an optimal allocation given the prices.

Step 4. Feasibility in Radner. The aggregate feasibility of allocation x in Radner follows from its feasibility in AD. To check for the feasibility of the asset demand, we have for each state s

$$\begin{aligned}\sum_i z_{ir^s} &= \sum_i \left(\sum_{s\in S} p_s \cdot x_{is} - \sum_{s\in S} p_s \cdot \omega_{is}\right) \\ &= \sum_{s\in S} p_s \cdot \left(\sum_i x_{is} - \sum_i \omega_{is}\right) = 0,\end{aligned}$$

by the strict aggregate feasibility.

Hence, we found a Radner equilibrium.

Direction 2) $\to$ 1). Suppose that we have a Radner equilibrium $(x, p, z, q, \tilde{q})$ form a Radner equilibrium with Arrow-Debreu securities for some asset demands $z \in \mathbb{R}^{I\mathcal{R}_{AD}}$ and $q \in \mathbb{R}^{\mathcal{R}_{AD}}, \tilde{q} \in \mathbb{R}^S$. We are going to find prices (p^*_{sl}) and check that with those prices, the allocation (x_{isl}) is an AD equilibrium.

Step 1. Prices. Define prices

$$p^*_{sl} = q_{r^s} p_{sl}.$$

Step 2. Budget constraints in AD. We check that allocation x_i satsifies the AD budget constraint:

$$\begin{aligned}p^* \cdot x_i - p^* \cdot \omega_i &= \sum_s \sum_l q_{r^s} p_{sl} (x_{isl} - \omega_{isl}) = \sum_s q_{r^s} \sum_l p_{sl} (x_{isl} - \omega_{isl}) \\ &\leq \sum_s q_{r^s} \sum_{r\in\mathcal{R}_{AD}} z_{ir} r_s = \sum_s q_s z_{ir^s} \leq 0,\end{aligned}$$

where the first inequality is due to the spot market ,and the second inequality from the asset market budget constraints of the Radner equilibrium. The last inequality implies that the allocation x_i satisfies the budget constraint under Arrow-Debreu prices p^* in period 0 market.

Step 3. Optimality in AD. We check that allocation x_i is optimal with respect to prices p^*. Suppose that $x_i'$is another allocation that satisfies the budegt constraint of problem (7.2)

$$p^* \cdot x_i' \leq p^* \cdot \omega_i$$

Let $z_{r^s}' = p_s \cdot x_{is}' - p_s \cdot \omega_{is}$ be an alternative demand for Arrow-Debreu security r^s. Then, such demands satisfies the budget constraints of problem (7.3):

- period 0 market:

$$\sum q_s z_{r^s}' = \sum_s q_s \sum_l p_{sl} \left(x_{isl}' - \omega_{isl}\right) = \sum_s \sum_l p_{sl}^* \left(x_{isl}' - \omega_{isl}\right) = p^* \cdot x_i' - p^* \cdot \omega_i \leq 0.$$

- Spot market s:

$$p_s \cdot x_{is}' - p_s \cdot \omega_{is} = z_{r^s}' = \sum_{r \in \mathcal{R}_{AD}} z_r' r_s.$$

Hence, choice (x', z') satisfies the budget constraints of problem (7.3) and, it is available for the Radner consumer. It follows that the utility of x^* cannot be higher than the utility of what the consumer actually chose, i.e., x.

Step 4. Feasibility in AD. The aggregate feasibility of allocation x in AD follows from its feasibility in Radner. Hence, $(x, p^*)$is an ADE. □

Comments:

- The Proposition says that we can find Radner allocations, by looking for AD instead, and vice versa.
- More importantly, the Proposition says that the Radner equilibrium with all the Arow-Debreu assets available for trade in period 0 market
- Convince yourself that you understand how the role of the the particular structure of the AD securities in the proof.

7.6. **(In)complete markets.** We can consider a generalization of the above setup. Let $\mathcal{R}$ be a finite set of assets.

Definition 10. We say that markets are complete (or more precisely, that the structure of assets $\mathcal{R}$ is complete) if if the collection of vectors $\mathcal{R}$ spans $\mathbb{R}^S$. Markets are incomplete, if they are not complete.

Given an arbitrary (i.e., complete or incomplete) asset structure, we can define the Radner equilibrium in the same way as in the Definition 9.

It turns out that (Proposition 19.E.2 from MWG) that if the markets are complete, the thesis of Proposition 3 holds unchanged. The intuition is that using complete asset structure, we can "recreate" Arrow-Debreu securities: anything can be achieved using the AD securities, can be also achieved using different but complete asset collection. To give you some idea how it can be done, let us look at how to reconstruct prices and demands for the assets (equations (7.4) of the proof of the Proposition 3).

First, observe that there must be at least S linearly independent vectors r in $\mathcal{R}$. Assume that there are vectors $\rho^1, ..., \rho^S \in \mathcal{R}$ and create a $S \times S$ matrix $R = \left[\rho^1, ..., \rho^S\right]$ where the vectors are written vertically. Because the vectors are linearly independent, matrix R has an inverse $A = R^{-1} = [\alpha_s^r]$, where s corresponds to columns and r corresponds to rows. Because $R \times A = I$ (the identity matrix), we have that for each state s,

$$\sum_{r \in \{\rho^1, ..., \rho^S\}} \alpha_s^r r = r^s$$

is the reconstructed state s AD security.

In order to satisfy equations (7.4), we want to find prices $q_1, ..., q_S$ of the assets in $\left\{\rho^1, ..., \rho^S\right\}$ such that the prices of the resulting Arrow-Debreu securities are 1: for each $s \in S$,

$$\sum_{r \in \mathcal{R}} \alpha_s^r q_r = 1.$$

We can do it by a bit of linear algebra. The above equations correspond to a vector equation

$$Aq = \begin{bmatrix} 1 \\ 1 \\ ... \\ 1 \end{bmatrix},$$

or

$$q = A^{-1} \begin{bmatrix} 1 \\ 1 \\ ... \\ 1 \end{bmatrix} = R \begin{bmatrix} 1 \\ 1 \\ ... \\ 1 \end{bmatrix}.$$

Thus, we can define prices

$$q_s = \sum_{r \in \mathcal{R}} r_s.$$

(Here, q_s is the price of an asset $\rho^s \in \mathcal{R}$.)

Similarly, we can reconstruct demands for assets in $\left\{\rho^1, ..., \rho^S\right\}$ so that they correspond to the demands for AD securities described in equations (7.4).

7.7. **Example.** We are going to analyze the following example:

Example 13. Two agents, Alice and Bob, have expected utility preferences over two states with probabilities $\pi_s \in (0,1)$ and vN-M utility function over consumption bundles with two goods $u(x,y) = (xy)^\kappa$, where we assume that $\kappa \leq \frac{1}{2}$. The endowments are

$$\omega_{A0} = (2,1) = \omega_{B1}, \omega_{A1} = (1,1) = \omega_{B0}.$$

In other words, Alice in state 0 and Bob in state 1 has an extra unit of good 1.

7.7.1. *Radner with spot markets only.* The consumer problem in the Radner with spot markets (i.e. problem (7.1)) is

$$\max_{x_{isl}} \sum_{s,l} \pi_s \left(\prod_l x_{isl}\right)^\kappa$$
$$\text{st. } \sum_l p_{sl} x_{isl} \leq \sum_l p_{sl} \omega_{isl} \text{ for each } s.$$

The utilities are strongly monotone and we have the Walras Law. Because of strict convexity, we can find a solution using the FOC. Let λ_i^s be the Lagrangian multiplier associated with the budget constraint of agent i on spot market s. The FOCs are

$$\pi_s \kappa \frac{\left(\prod_l x_{isl}\right)^\kappa}{x_{isl}} = \lambda_i^s p_{sl} \text{ for each } i, s, l,$$

or

$$x_{isl} = \frac{\pi_s \kappa \left(\prod_l x_{isl}\right)^\kappa}{\lambda_i^s} \frac{1}{p_{sl}} = \alpha_i^s \frac{1}{p_{sl}},$$

where we use α_i^s to simplify the notation. We can derive the value of α_i^s from the budget constraint:

$$\sum_l p_{sl} \omega_{isl} = \sum_l p_{sl} x_{isl} = \sum_l p_{sl} \alpha_i^s \frac{1}{p_{sl}} = 2\alpha_i^s.$$

Hence,

$$x_{isl} = \frac{1}{2}\frac{1}{p_{sl}}\sum_s p_{sl}\omega_{isl}.$$

Thus, in state 0, the demands for each good l are

$$x_{Al0}\left(p_1, p_2\right) = \frac{2p_1 + p_2}{2p_l}, x_{Bl0}\left(p_1, p_2\right) = \frac{p_1 + p_2}{2p_l}.$$

(Knowing that the state-independent utility has a Cobb-Douglas form, you could have skipped these steps and go directly to solution that each agent will spend half of their wealth in each state on each good. This really corresponds to noticing that, because the utility and the budget constraints are separable, we can replace the Radner problem (7.1), by consumer optimization problems for each state, where in each state, the consumer is maximizing its vNM utility. The method above, i.e, to derive the demands directly from the Radner problem (7.1), is perhaps longer, as it does not rely on tricks, but it is solid, and it will always get you to the right spot.)

Next, we use the market clearing:

$$\frac{2p_1 + p_2}{2} + \frac{p_1 + p_2}{2} = p_1 \sum_i \omega_{i01} = 3p_1,$$
$$\frac{2p_1 + p_2}{2} + \frac{p_1 + p_2}{2} = p_2 \sum_i \omega_{i02} = 2p_2,$$

which determines the prices $p_1 = \frac{2}{3}p_2$.

Hence, the final allocations are

$$x_{AX0} = \left(\frac{7}{4}, \frac{7}{6}\right), x_{BX0} = \left(\frac{5}{4}, \frac{5}{6}\right).$$

Similarly, for state $s = 1$, we get

$$x_{AX1} = \left(\frac{5}{4}, \frac{5}{6}\right), x_{BX1} = \left(\frac{7}{4}, \frac{7}{6}\right).$$

Alice is happier in state 0, Bob in state 1.

Notice that the probabilities of states are not important for the calculations (which are done state-by-state).

7.7.2. *Arrow-Debreu equilibrium.* The consumer problem in the AD problem (i.e. problem (7.2)) is

$$\max_{x_{isl}} \sum_{s,l} \pi_s \left(\prod_l x_{isl} \right)^{\kappa}$$

$$\text{st. } \sum_{s,l} p_{sl} x_{isl} \leq \sum_{s,l} p_{sl} \omega_{isl}.$$

The utilities are strongly monotone and we have the Walras Law. Because of strict convexity, we can find a solution using the FOC. Let λ_i be the Lagrangian multiplier associated with the budget constraint of agent i (notice that we have only one multiplier and one constraint now.) The FOCs are

$$\pi_s \kappa \frac{\left(\prod_l x_{isl}\right)^{\kappa}}{x_{isl}} = \lambda_i p_{sl} \text{ for each } i, s, l,$$

or

$$x_{isl} = \frac{\pi_s \kappa \left(\prod_l x_{isl}\right)^{\kappa}}{\lambda_i} \frac{1}{p_{sl}}.$$

We want to figure out λ_i. Notice that we can multiply the above equations across goods l to obtain

$$\prod_l x_{isl} = \frac{(\kappa \pi_s)^L}{\lambda_i^L} \left(\prod_l x_{isl} \right)^{\kappa L} \frac{1}{\prod_l p_{sl}}.$$

Thus, after some algebra,

$$\left(\prod_l x_{isl} \right)^{\kappa} = (\kappa \pi_s)^{\frac{\kappa L}{1-\kappa L}} \lambda_i^{-\frac{\kappa L}{1-\kappa L}} \left(\prod_l p_{sl} \right)^{-\frac{\kappa}{1-\kappa L}}.$$

Substituting back to the demand for -contingent good sl, we obtain

$$x_{isl} = (\kappa \pi_s)^{\frac{1}{1-\kappa L}} \lambda_i^{-\frac{1}{1-\kappa L}} \left(\prod_{l'} p_{sl'} \right)^{-\frac{\kappa}{1-\kappa L}} \frac{1}{p_{sl}}.$$

Now, we can derive the value of $\lambda_i^{-\frac{1}{1-\kappa L}}$ from the budget constraint:

$$\sum_{s,l} p_{sl} \omega_{isl} = \sum_{s,l} p_{sl} x_{isl} = \sum_{s,l} p_{sl} (\kappa \pi_s)^{\frac{1}{1-\kappa L}} \lambda_i^{-\frac{1}{1-\kappa L}} \left(\prod_{l'} p_{sl'} \right)^{-\frac{\kappa}{1-\kappa L}} \frac{1}{p_{sl}}$$

$$= \lambda_i^{-\frac{1}{1-\kappa L}} 2 \sum_s (\kappa \pi_s)^{\frac{1}{1-\kappa L}} \left(\prod_{l'} p_{sl'} \right)^{-\frac{\kappa}{1-\kappa L}},$$

which implies that $\lambda_i^{-\frac{1}{1-\kappa L}} = \frac{1}{2}\left(\sum_s (\kappa\pi_s)^{\frac{1}{1-\kappa L}} \left(\prod_l p_{sl}\right)^{-\frac{\kappa}{1-\kappa L}}\right)^{-1}\left(\sum_{s,l} p_{sl}\omega_{isl}\right)$. Thus, tadaam, we get our demand equations:

$$x_{isl} = \frac{1}{2}\frac{(\kappa\pi_s)^{\frac{L}{1-\kappa L}}\left(\prod_{l'} p_{sl'}\right)^{-\frac{\kappa}{1-\kappa L}}}{\sum_{s'}(\kappa\pi_{s'})^{\frac{1}{1-\kappa L}}\left(\prod_{l'} p_{s'l'}\right)^{-\frac{\kappa}{1-\kappa L}}}\frac{1}{p_{sl}}\left(\sum_{s',l'} p_{s'l'}\omega_{is'l'}\right).$$

$$\frac{1}{2}\frac{\left(\prod_{l'}\frac{1}{\pi_s}p_{sl'}\right)^{-\frac{\kappa}{1-\kappa L}}}{\sum_{s'}\left(\prod_{l'}\frac{1}{\pi_s}p_{sl'}\right)^{-\frac{\kappa}{1-\kappa L}}}\frac{1}{p_{sl}}\left(\sum_{s',l'} p_{s'l'}\omega_{is'l'}\right).$$

$$= \alpha_s(p)\frac{1}{p_{sl}}w_i,$$

where we use the last equality to emphasize that each consumer consumes exactly the same vector of state-contingent goods up to scalar. The scalar is equal to the individual wealth w_i.

(Again, you could have used shortcuts and derived this fact directly from the fact that the consumer is maximizing homothetic[25] and strictly convex preferences. Given the amount of mistakes I did when taking shortcuts, I prefer the long, but safe way.)

We can move to market clearing:

$$\alpha_s(p)\frac{1}{p_{sl}}\left(\sum_i w_i\right) = \sum_i \omega_{isl} = \omega_{sl}. \tag{7.5}$$

By dividing the above equations for goods $l=1$ by an equation for good $l=2$ keeping the state s constant, we obtain

$$\frac{p_{s2}}{p_{s1}} = \frac{\omega_{s1}}{\omega_{s2}},$$

or that $p_{s2} = \frac{\omega_{s1}}{\omega_{s2}}p_{s1}$. We have

$$\alpha_s(p)\sum_i w_i = \frac{1}{2}\frac{\left(\frac{1}{\pi_s^2}p_{s1}^2\frac{\omega_{s1}}{\omega_{s2}}\right)^{-\frac{\kappa}{1-\kappa L}}}{\sum_{s'}\left(\frac{1}{\pi_s^2}p_{s\prime 1}^2\frac{\omega_{s\prime 1}}{\omega_{s\prime 2}}\right)^{-\frac{\kappa}{1-\kappa L}}}\left(2\sum_s p_{s1}\omega_{s1}\right).$$

[25] The utility $\sum_{s,l}\pi_s\left(\prod_l x_{isl}\right)^{\kappa}$is not homothetic, but it corresponds to homothetic preferences: we can turn it tino homothetic througha monotonic transformation $\left(\sum_{s,l}\pi_s\left(\prod_l x_{isl}\right)^{\kappa}\right)^{\frac{1}{\kappa}}$.

By dividing equattion (7.5) for state $s = 0$ by state $s = 1$, and keeping the good $l = 1$, we obtain

$$\frac{\left(\frac{1}{\pi_0^2} p_{01}^2 \frac{\omega_{01}}{\omega_{02}}\right)^{-\frac{\kappa}{1-\kappa L}}}{\left(\frac{1}{\pi_1^2} p_{11}^2 \frac{\omega_{11}}{\omega_{12}}\right)^{-\frac{\kappa}{1-\kappa L}}} \frac{p_{11}}{p_{01}} = \frac{\omega_{01}}{\omega_{11}},$$

or

$$\frac{\omega_{01}}{\omega_{11}} = \left(\frac{p_{11}}{p_{01}}\right)^{\frac{1}{1-2\kappa}} \left(\frac{\omega_{02}}{\omega_{01}} \frac{\omega_{11}}{\omega_{12}}\right)^{\frac{\kappa}{1-2\kappa}} \left(\frac{\pi_1}{\pi_0}\right)^{-2\frac{\kappa}{1-2\kappa}},$$

or

$$\frac{p_{11}}{p_{01}} = \left(\frac{\omega_{01}}{\omega_{11}}\right)^{1-2\kappa} \left(\frac{\omega_{01}}{\omega_{11}} \frac{\omega_{12}}{\omega_{02}}\right)^{\kappa} \left(\frac{\pi_1}{\pi_0}\right)^{2\kappa} = \left(\frac{\omega_{01}}{\omega_{11}}\right)^{1-\kappa} \left(\frac{\omega_{12}}{\omega_{02}}\right)^{\kappa} \left(\frac{\pi_1}{\pi_0}\right)^{2\kappa}.$$

Together with equations $p_{s2} = \frac{\omega_{s1}}{\omega_{s2}} p_{s1}$, this determines the prices (up to multiplicative constant). Substituting back to the demand equation, we can compute the allocation.

7.7.3. *Radner with Arrow-Debreu assets.* We are going to use the proof of the Proposition to find the prices and allocations of Radner equilibrium that correspond to the Arrow-Debreu equilibrium that we found in Example **??** .

7.8. **Non-expected utility theories.** Non-expected utilities are possible. However, there is a problem: Non-expected utility theories, like ambiguity aversion for example, are plagued by the dynamic inconsistency. That means is that what is optimal choice in period 1 may be different than what is optimal choice once period 1 happens. The dynamic inconsistency creates a problem of how to approach the period 1 markets (ie. the spot markets).

Example 14. TBA.