5. Welfare Theorems[19]

The Welfare theorems relate Pareto-efficiency and Walrasian equilibrium.

- The First Welfare Theorem (FWT) says that all Walrasian equilibrium allocations are Pareto-optimal. This is very important result because it says that, in Walrasian equilibrium, we cannot improve anybody's life without reducing somebody's else.
- In general, there are many Pareto-optimal allocations, and they differ in important aspects, like the distribution of utility or wealth. One might be concerned whether some FWT does not say how equitable are the equilibrium allocations. The Second Welfare Theorem says that, if we are allowed to transfer wealth (or endowment), then by appropriately choosing transfers, we can implement any Pareto-optimal allocation. An important assumption needed for the SWT is that the preferences are convex.
- The FWT has an interesting corollary that is an important in its own. Consider an exchange economy with privately owned endowments. An equilibrium requires possible participation of all agents in the economy, whose equilibrium allocations may differ from their endowments (because each one of the trades their goods). Suppose that a subset of the agents would like to remove themselves from the general economy and reallocate their privately owned goods between each other. Can they be better off by doing so? It turns out, not.

Together, the Welfare Theorems are theoretical underpinnings for the argument the "free-market", possibly with wealth-correcting transfers, is better left untouched.

5.1. **First Welfare Theorem.**

Theorem 6. *Let (x, p) be a Walrasian equilibrium such that all prices are non-negative, $p > 0$, and the Walras Law holds for each consumer (for instance, because (a) for each good, at least one consumer is strictly monotonic in this good's direction, or (b) prices are strictly positive, $p \gg 0$ and preferences are locally non-satiated). Then, x is Pareto-optimal.*

[19] $November 12, 2019$.

The two assumptions: local non-satiation and strictly positive prices, are required to handle the problem of "bads" (see the discussion in Remark 1). Notice that due to Lemma 2, the assumptions are satisfied if, the preferences are strongly monotone.

The assumptions (local non-satiation) do not allow for indivisble goods. This can be fixed if we allow the members to trade lotteries over goods.

Proof. Suppose that (x, p) is a Walrasian equilibrium that is not Pareto-optimal. In particular, there is a feasible allocation x^* such that for each consumer i, $u_i(x_i) \leq u_i(x_i^*)$, and there is a consumer j such that the inequality is strict.

For each consumer i, if x_i^* is individually feasible and $u_i(x_i) \leq u_i(x_i^*)$, it must be that $x_i^* \in x_i^*(p, w_i)$ is a part of the demand. The Walras Law implires that

$$p \cdot x_i^* \geq p \cdot x_i.$$

Moreover, it must be that

$$p \cdot x_j^* > p \cdot x_j.$$

Otherwise, because $x^*$is feasible, x_j^* is individually feasible, hence available for consumer j, and as it is strictly better, consumer j should not have chosen x_j. Finally, notice that profit maximization implies that

$$p \cdot y_j^* \leq p \cdot y_j.$$

The Walras Law applied at the equilibrium allocation implies that for each consumer i

$$p \cdot x_i = p \cdot \omega_i + \sum_j \theta_{ij} p \cdot y_j.$$

By summing up across consumers we obtain the aggregate budget balance condition

$$p \cdot \sum_i x_i = p \cdot \omega + p \cdot \sum_j y_j.$$

The above inequalities imply that

$$p \cdot \sum_i x_i^* > p \cdot \sum_i x_i = p \cdot \omega + p \cdot \sum_j y_j \geq p \cdot \omega + p \cdot \sum_j y_j^*.$$

Because all prices are non-negative, there must be a good l such that

$$x_i^{*,l} > \omega_i^l + \sum_j y_j^{*,l}.$$

But this contradicts feasibility of allocation x^*. □

Exercise 10. Suppose that we use a stronger definition of Walrasian equilibrium, with strict instead of weak aggregate feasibility. In such a case, show that the FWT does not need the assumption that prices are non-negative.

5.2. **Separating hyperplane theorem.** For the proof of the Second Welfare Theorem, we will need the Separating Hyperplane Theorem.

Theorem 7. *Let A and B be two disjoint nonempty convex subsets of $\mathbb{R}^n$. Then there exist a nonzero vector $p \in \mathbb{R}^n$ such that*

$$\sup_{a \in A} p \cdot a \leq \inf_{b \in B} p \cdot b.$$

Of course, the convexity is very important for the Theorem. An infinite generalization: Hahn-Banach theorem.

5.3. **Equilibrium with transfers.** Instead of initial endowments, we can also consider original endowments and transfers.

Definition 4. A Walrasian equilbrium with transfers is a feasible allocation $x = (x_1, ..., y_J)$, a vector of prices p, and transfers $(\tau_1, ..., \tau_I)$ such that $\sum_i \tau_i = 0$ such that

- for each i, $x_i \in x^* (p, w_i)$, where $w_i = p \cdot \omega_i + \sum_j \theta_{ij} (p \cdot y_j) + \tau_i$,
- for each j, $y_j \in y^* (p)$.

Importantly, the transfers do not have a form of taxes, in many ways:

- they differ between individuals,
- they do not depend on behavior: they are not consumption taxes nor income taxes,
- they are allocated by omniscient Walrasian taxmen.

5.4. **Second Welfare Theorem.** Suppose that x^* is a Pareto-optimal allocation. Recall that consumption plan $x = (x_1, ..., x_I)$ Pareto-dominates x^* (we will write $x^* \prec_P x$) if $x_i^* \preceq_i x_i$ for each i, with at least one inequality strict.

Theorem 8. *(SWT) Suppose that (a) all preferences are convexand all technologies are convex, and (b) preferences are locally non-satiated and continuous,, the endowments are interior, technologies satisfy the possibility of inaction.*
Then, for each Pareto-optimal allocation x^*, there exists a vector of non-zero, non-negative prices p^* and transfers τ^* such that (x^*, p^*, τ^*) is a Walrasian equilibrium with prices.

The first set of assumptions is substantive. Convexity is important. Example in Edgeworth box.

The second one is technical and we use it in similar way as in the proof of the existence of Walrasian Equilibrium (Theorem 4).

Regarding the value of the SWT: the Walrasian Taxmen who has all the information needed to figure out the level of taxes necessary to guarantee a given level of equitability of the allocation, probably would have also information that is necessary to compute the Pareto-efficient allocation, without a need for any market mechanism.

5.5. **Proof of the SWT.** Let $x^* = (x_1^*, ..., x_I^*, ..., y_J^*)$ be a Pareto-optimal allocation. Let $\tilde{x} = \sum_i x_i^*$ be the aggregate demand in this allocation.

The proof is divided into a sequence of steps.

5.5.1. *Aggregate demands and supplies.* We define sets:

- the set of aggregate demands in consumption plans that Pareto-dominate x^*:
$$X^* = \left\{ \sum x_i : x \in \prod X_i \text{ st. } x^* \prec_P x \right\}.$$
- the set of "available" aggregate consumptions
$$Y_0 = \omega + \sum_i Y_i, \text{ and}$$
$$Y^* = \{x : x \leq y \text{ for some } y \in Y_0\}.$$
- Observe that, by the aggregate feasibility of the Pareto allocation x^*, we have $\tilde{x} \in Y^*$.

We have two simple Lemmas:

Lemma 8. *X^* and $Y^*$are disjoint. Also, $\tilde{x} \in Y^*$ and $\tilde{x} \in clX^*$.*

Proof. On the contrary, suppose that there is $z \in X^* \cap Y^*$. Then, there is a consumption plan $(x_1, ..., x_I)$ that Pareto dominates x^* and such that $\sum x_i = z$. Moreover, there is a production plan $(y_1, ..., y_J)$ such that $y_j \in Y_j$for each j and

$$z \leq \omega + \sum y_j.$$

Consider an allocation $(x_1, ..., x_I, y_1, ..., y_J)$. This allocation is individually and aggregate feasible. It also Pareto-dominates the original allocation. Hence, x^* is not Pareto-efficient. Contradiction shows the first claim.

For the second claim, notice that $\tilde{x} \in Y^*$ because $x^*$is aggregate feasible. Local non-satiation implies that there is a sequence of consumption bundles $x_i^n \to x_i^*$ for each i such that $x_i^* \prec x_i^n$. □

Lemma 9. *$X^*$and $Y^*$are convex.*

Proof. The convexity of Y_0 follows from Exercise 1. The convexity of Y^* is left as an exercise.

For the convexity of X^*, take any $x, x' \in X^*$ and $\alpha \in (0, 1)$. Let i be a player such that $x^* \prec_i x$. Hence, $x_j^* \preceq_j x_j, x_j'$for each j, $x_i^* \prec_i x_i$, and $x_i^* \preceq_i x_i'$. By the convexity of preferences,

$$x_j^* \preceq_j \alpha x_j + (1 - \alpha) x_j',$$
$$x_i^* \prec_i \alpha x_i + (1 - \alpha) x_i'.$$

It follows that $x^* \prec_P \alpha x + (1 - \alpha) x'$. □

5.5.2. *Prices.* The Separating Hyperplane Theorem and Lemmas 8 and 9 implies that that there exists a non-zero $p \in \mathbb{R}^n$ such that

$$\sup_{y \in Y^*} p^* \cdot y \leq \inf_{x \in X^*} p^* \cdot x. \tag{5.1}$$

Note that the prices must be positive; the reason is that set $Y^*$is unbounded from below, and if one price $p_l < 0$, then we can find $y_n = (y_1, ..., y_{l,n}, ..., y_L) \in Y^*$ such that $y_{n,l} \to -\infty$ and $p^* \cdot y_n \to \infty$, which would contradict the above inequality.

Because of the second part of Lemma 8, we have

$$p^* \cdot \tilde{x} \leq \sup_{y \in Y^*} p^* \cdot y \text{ and } \inf_{x \in X^*} p^* \cdot x \leq p^* \cdot \tilde{x},$$

which, together with (5.1) implies that

$$\sup_{y\in Y^*} p^*\cdot y = p^*\cdot\tilde{x} = \inf_{x\in X^*} p^*\cdot x. \tag{5.2}$$

5.5.3. *Transfers.* We define transfers

$$\tau_i^* = p^*\cdot x_i^* - p^*\cdot\left(\omega_i + \sum_j \theta_{ij}\left(p^*\cdot y_j^*\right)\right).$$

These transfers make sure that the wealth of agent i is $w_i^* = p\cdot x_i^*$.

Because $x^*$is feasible, this concludes the proof of the existence of Walrasian equilibrium with transfers.

5.5.4. *Firms.* We show that each firm is maximizing profits: $y_j^*\in\arg\max_{y\in Y_j} p^*\cdot y$. If not, then there is a production plan $(y_1,...,y_J)\in\prod_j Y_j$ such that$p\cdot y_j\geq p\cdot y_j^*$, with at least one inequality strict. But then, let $y=\omega+\sum_j y_j$. We have

$$p^*\cdot y = p\cdot\omega+\sum_j p\cdot y_j > p\cdot\omega+\sum_j p\cdot y_j^* = p\cdot\left(\omega+\sum_j y_j^*\right)\geq p\cdot\tilde{x},$$

where the last inequality comes from the aggregate feasibility (i.e., $\tilde{x}\leq\omega+\sum_j y_j^*$) and the fact that the prices must be positive (see Step 1).

5.5.5. *Consumers.* Finally, we show that each consumer i is optimizing at p^*. On the other hand, suppose that for some player i, there is x_i such that $x_i^*\prec x_i$ and $p\cdot x_i\leq p\cdot x_i^*$. By Lemma 6 and the continuity of preferences, there exists $x_i'\in X_i$ such that $x_i^*\prec x_i'$ and $p\cdot x_i<p\cdot x_i^*$. Let

$$x=\left(x_1^*,..,x_i',...,x_I^*\right).$$

Then, $x^*\prec_P x$, which implies that $\sum_{j\neq i} x_j^*+x_i'\in X^*$. But,

$$p\cdot\left(\sum_{j\neq i} x_j^*+x_i'\right) < p\cdot\left(\sum_{j\neq i} x_j^*+x_i^*\right) = p\cdot x^*,$$

which contradicts the second inequality in (5.2).

5.6. **Core.** There are other ways of selecting a feasible allocation in the economies apart from the Walrasian equilibrium. One of them is a core.

In this subsection, we assume that we have a pure exchange economy.

Definition 5. Say that a feasible allocation x can be blocked by coalition $S \subseteq I$ if there is an "$S-$allocation" $(x_i')_{i\in S} \in \prod_{i\in S} X_i$ that (a) satisfies (weak) S- aggregate feasibility:

$$\sum_{i\in S} x_i' \leq \sum_{i\in S} \omega_i,$$

and such that (b) for each $i \in S$, $u_i(x_i') > u_i(x_i)$. A core is a set of feasible allocations in the whole economy that cannot be blocked by any coalition.

To understand this definition, think about group of agents S removing themselves from the global economy, and reallocating their private endowments. An original allocation can be blocked, if there is a reallocation that improves the utility of every single agent in the group.

Exercise 11. Show that under the hypothesis of Proposition 1,

Theorem 9. *An allocation in a Walrasian equilibrium with non-negative prices (i.e., $p > 0$) belongs to the core.*

Proof. The proof follows the proof of the First Welfare Theorem. Suppose that (x, p) is a Walrasian equilibrium. Suppose that there is an "$S-$allocation" $(x_i')_{i\in S} \in \prod_{i\in S} X_i$ that (a) satisfies (weak) S- aggregate feasibility:

$$\sum_{i\in S} x_i' \leq \sum_{i\in S} \omega_i,$$

and such that $u_i(x_i') > u_i(x_i)$. Because of the latter, it must be that none of the bundles x_i' is available at prices p:

$$p \cdot x_i' > p \cdot \omega_i.$$

By summing over $i \in S$, we get

$$\sum_{i\in S} p \cdot x_i' > \sum_{i\in S} p \cdot \omega_i.$$

But this contradicts S-feasibility. □

Remark 2. The proof and assumptions of Theorem 9 are very similar to the proof and assumptions of Theorem 6. Here, we do not require the Walras Law - the reason is that the definition of blocking allocation is stronger than the Pareto-improvement: we require that each member of the coalition is strictly better off (this ensures that the blocking allocation must be strictly outside of the budget for each consumer; something that we would not necessarily have in the proof of Theorem 6 without the Walras Law).

Converse does not hold. Picture in 2×2 economy, Edgeworth box (18B1 of MWG): show that there can be continuum of core allocations.

5.7. **Core convergence.** TBA.