LAST (Family) NAME: ______________________________

FIRST (Given) NAME: ______________________________

STUDENT NUMBER: ______________________________

UNIVERSITY OF TORONTO

Faculty of Arts and Science

DECEMBER 2019 EXAMINATIONS

ECO2020 part II, General Equilibrium

Instructor: Marcin Pęski

Duration - 120 minutes

No Aids Allowed

Exam Reminders:

- Fill out your name and student number on the top of this page.
- Do not begin writing the actual exam until the announcements have ended and the Exam Facilitator has started the exam.
- Your answer must fit into the space provided. Use a PENCIL. Ink can't be erased if you change your mind.
- Two last pages contain the space for the rough-work. If you must use the rough-work pages for an answer, indicate this clearly by writing "Continued on last page".
- There are four questions with total worth of 100 points. Each question has the same value.
- You need to provide arguments for each answer.
- If you cannot solve one part of the problem, don't give up and try to solve the next one.
- If the question explicitly asks you to prove a result from the class, you must carefully describe the proof. Otherwise, you may use any result from the class given that you clearly state the assumptions, thesis and verify that the assumptions hold in your application.
- An advice: It is always a good idea to start each "proof" answer with writing precisely what is that you want to show. It will help you to make it precise when thinking about solution. It will also allow me to give you a tiny bit of partial credit if the rest of the answer turns out to be wrong.

Good luck!

Students must hand in all examination materials at the end.

1. (25) Consider the following social choice problem. A society collectively chooses an action $a \in A$ and a system of transfers τ_i to each individual. The transfers have to satisfy the feasibility condition $\sum \tau_i \leq 0$. An allocation is a pair (a, τ) of $a \in A$ and feasible transfers. Agent i's utility from the allocation is

$$u_i(a) + k_i \tau_i,$$

where $k_i > 0$ is some agent-specific constant.

(a) (5) State a definition of a Pareto-optimal allocation appropriate for this environment.

Solutions: Allocation (a, τ) is *Pareto-optimal* if for any other (b, θ) such that $\sum \theta_i \leq 0$, if $u_i(b) + k_i \theta_i > u_i(a) + k_i \tau_i$ for some agent, then there is an agent $j \neq i$ such that $u_j(b) + k_j \theta_j < u_j(a) + k_j \tau_j$.

(a) (20) Show (using the definition of Pareto-optimality) that if an allocation (a, τ) is Pareto-optimal, then $a \in \arg\max \sum_i \frac{1}{k_i} u_i(a)$. *Solutions*: Define welfare as

$$W(a) = \sum_i \frac{1}{k_i} u_i(a).$$

We show first that if (a, τ) is Pareto-optimal, then it must maximize welfare. Suppose that $b \in A$ is such that $W(a) < W(b)$. We are going to create a feasible allocation (b, θ) that Pareto-dominates (a, τ). Indeed, for each i, let

$$\theta_i = \tau_i + \frac{1}{k_i}(u_i(a) - u_i(b)) + \frac{1}{I}[W(b) - W(a)].$$

The individual utility from the new allocation is equal to

$$\begin{aligned} u_i(b) + k_i\theta_i &= u_i(b) + k_i\tau_i + u_i(a) - u_i(b) + \frac{1}{I}k_i[W(x') - W(x)] \\ &= u_i(a) + k_i\tau_i + \frac{1}{I}[W(b) - W(a)] > u_i(a) + k_i\tau_i. \end{aligned}$$

To check that the allocation is feasible, we only need to look at the aggregate feasibility of the numeraire (because nothing else changed). But,

$$\begin{aligned} \sum_i \theta_i &= \sum_i \tau_i + \sum_i \frac{1}{k_i}(u_i(a) - u_i(b)) + I\frac{1}{I}[W(x') - W(x)] \\ &= \sum_i \tau_i + W(a) - W(b) + [W(b) - W(a)] \\ &= \sum_i \tau_i. \end{aligned}$$

Hence, (b, θ) is feasible and it Pareto-dominates (a, τ).

2. (25) A farmland is divided by the Deep River flowing east-west. The Deep River is crossed by the Aurora Highway going north-south. In order to irrigate farms on the northern side of the Deep River, a ditch needs to be built. The farms lie along the Highway, starting from farm 1 and ending with farm I and a ditch to farm $i \leq I$ will necessarily irrigate all farms $j \leq i$. The cost of running the ditch to farm i is equal to $c_i > 0$ and it is increasing in i. The value of a coalition of farmers S is defined as the negative of the total cost of running a ditch through all farms in S, or

$$v(S) = -\max_{i \in S} c_i.$$

From now on, we consider the irrigation problem (I, v) as the TU-cooperative game with characteristic function v.

   (a) (5) State the definition of a convex game. *Solutions*: A TU-game (I, v) is convex if for any two coalition $T, S \subseteq I$, we have

$$v(T) + v(S) \leq v(T \cap S) + v(T \cup S).$$

(b) (10) Show that the above defined game is convex. *Solutions*: Suppose w.l.o.g. that $v(T) \leq v(S)$, meaning that T has a farmer that lives farther away that any farmer in S. Then, $v(T \cap S) \geq v(S)$ (because S contains a farmer that lives (weakly) further away than any farmer in the intersection of S and T) and $v(T \cup S) = v(T)$. The inequality follows.

(c) (10) Suppose that the local authority proposes to build a ditch to the last farm I. To pay for it, the authority proposes to charge taxes

$$t_1 = c_1,$$
$$t_2 = c_2 - c_1,$$
$$t_3 = c_3 - c_2,$$
$$...$$
$$t_I = c_I - c_{I-1}.$$

Let $x_i = -t_i$ be the utility vectors associated with the above taxes. Is x in the core of the game? Justify your answer without relying on any results from the class. *Solutions*: To

check that the payments are in the core of the game, it is enough to show that for each colaition S,

$$-\sum_{i \in S} t_i \geq v(S).$$

But

$$-\sum_{i \in S} t_i = \sum_{i \in S} c_{i-1} - c_i \geq \sum_{i : i \leq \max_{j \in S} j} c_{i-1} - c_i = c_{\max_{j \in S} j} = v(S).$$

(d) (5) Is there any other core allocation? *Solutions*: Yes, for instance, $t_I = v(I)$ and $t_i = 0$ for $i < I$.

3. (25) Consider an exchange economy, with two consumers $i = A, B$ who trade on two markets $m = S, T$. There are 8 goods $1, 2..., 8$; goods $1...., 4$ are traded on market S and goods $5, ..., 8$, are traded on market T. The consumers have identical preferences

$$u(x_{i1}, ..., x_{i8}) = (x_{i1} \cdot x_{i2} \cdot ... \cdot x_{i8})^{1/8}.$$

Suppose that initial endowments of the two goods are equal to

$$\omega_A = (2, 4, 6, 8, 10, 8, 3, 8),$$
$$\omega_B = (3, 6, 9, 12, 10, 8, 3, 8).$$

(a) (5) Find an allocation in a Radner equilibrium. (Hint: I am not asking for the equilibrium prices.) *Solutions*: Because preferences are homothetic, identical, strongly monotone and convex, players demands are co-linear. Because their endowments are co-linear, and B is 50% richer in market S and the same rich in market T, her demand in market S is 50% larger than the demand of A for goods 1234 and and her demand for 5678 is the same as Bs. Hence,

$$x_i = \omega_i.$$

(b) (10) Explain that the allocation that you found in previous question is not Pareto-optimal. Give an example of a Pareto-improving allocation. *Solutions*: The allocation is not Pareto-optimal. Observe that

$$u(x_B) = \left(\frac{3}{2}\right)^{1/2} u(x_A).$$

Let $\alpha_i, \beta_i > 0$ be constants and consider allocations

$$y_i^{\alpha,\beta} = (y_{iS}, y_{iT}) = (\alpha_i x_{iS}, \beta_i x_{iT}).$$

The utility from such allocations is equal to

$$u_i\left(y_i^{\alpha,\beta}\right) = \alpha_i^{1/2}\beta_i^{1/2},$$

so, as long as $\alpha_i\beta_i > 1$, the allocation is an improvement. Additionally, allocation $y^{\alpha,\beta}$ is feasible if

$$y_{AS} + y_{BS} = \left(\alpha_A + \frac{3}{2}\alpha_B\right)x_{AS} = \omega_S = \left(1 + \frac{3}{2}\right)x_{AS}, \text{ or}$$
$$\alpha_A + \frac{3}{2}\alpha_B = \frac{5}{2}.$$

Suppose that $\alpha_A = 1 + \varepsilon, \alpha_B = 1 - \frac{2}{3}\varepsilon, \beta_A = 1 - \gamma, \beta_B = 1 + \gamma$. Then, we need

$$1 < (1+\varepsilon)(1-\gamma) \text{ or } \varepsilon - \gamma - \varepsilon\gamma > 0,$$
$$1 < \left(1 - \frac{2}{3}\varepsilon\right)(1+\gamma) \text{ or } \gamma - \frac{2}{3}\varepsilon - \varepsilon\gamma > 0.$$

Suppose that $\gamma = \frac{5}{6}\varepsilon$ are very very small. Then, both inequalities are satisified.

(c) (10) Assume again that goods are traded only on their own respective markets. However, suppose that there is another market F, on which a single financial asset can be traded. The asset can be traded in both positive and negative amounts. Each unit of a financial asset r delivers r_S units of currency in market S and $-r_T$ in market T, where $r_S, r_T > 0$. The currency later can be exchanged for goods. Find a Radner equilibrium of the economy in which the asset plays non-trivial role. Show that the equilibrium allocation Pareto improves on the allocation that you found in part 1. (Your answer may depend on a numerical calculation. In such a case, if you don't have a calculator and cannot do it by hand - don't worry about it, just clearly describe the objective as an outcome of some calculation.) *Solutions:* Consider an equilibrium with three markets. Because there is a single asset traded in market F, then if the asset has any role, and the demands are not zero, then the budget constraint in market F imlies that the asset must have price 0. The good allocations must remain the fractions of the endowments on each market; hence, they must be co-linear with the allocations found in part 1. In particular, up to normalization, the good prices on the markets S and T must be exactly the same as be exactly the same as in part 1. Suppose that z_i is the demand for asset r by customer i; 1 unit of money on market S buys q_S bundles x_{AS} on market S and 1 unit of money on market T buys q_T units of bundle x_{AT} on market T. Suppose that the asset returns are $(r_A, -r_B)$. The allocations in the Radner equilibrium are equal to:

$$y_A = (y_{AS}, y_{AT}) = (x_{AS}(1 + z_A q_S r_S), x_{AT}(1 - z_A q_T r_T)),$$
$$y_B = (y_{BS}, y_{BT}) = \left(x_{AS}\left(\frac{3}{2} + z_B q_S r_S\right), x_{AT}(1 - z_B q_T r_T)\right)$$

where r are some constants.

Because the price of teh asset is 0, each agent is free to choose demand to maximize its utility. Consumer A's utility is

$$u(y_A) = u(x_A) \cdot (1 + z_A q_S r_S)^{1/2} (1 - z_A q_T r_T)^{1/2}.$$

The FOC imply that

$$q_S r_A (1 - z_A q_T r_B) = q_T r_B (1 + z_A q_S r_S),$$

or

$$z_A = \frac{q_S r_S - q_T r_T}{2 q_S r_S q_T r_T}.$$

Consumer B's utility is

$$u(y_B) = u(x_B) \cdot \left(1 + \frac{2}{3} z_B q_S r_S\right)^{1/2} \left(1 - z_B q_T r_T\right)^{1/2}.$$

The FOC:

$$q_S r_A \left(1 - z_B q_T r_T\right) = q_T r_B \left(\frac{3}{2} + z_B q_S r_S\right),$$

or

$$z_B = \frac{q_S r_S - \frac{3}{2} q_T r_T}{2 q_S r_S q_T r_T}.$$

The feasibility on the financial market implies that

$$q_S r_S - q_T r_T + q_S r_S - \frac{3}{2} q_T r_T = 0,$$

or

$$2 q_S r_S = \frac{5}{2} q_T r_T.$$

This can be satisfied by proper choice of q_S, q_T for any r_A, r_B.

Let $q_S r_A = \lambda$; then, $q_T r_T = \frac{4}{5}\lambda$. The demands are equal to

$$z_A = \frac{\lambda - \frac{4}{5}\lambda}{2\frac{4}{5}\lambda^2} = \frac{1}{8\lambda}, z_B = \frac{\lambda - \frac{6}{5}\lambda}{2\frac{4}{5}\lambda^2} = -\frac{1}{8\lambda}$$

Then, substituting to the equations for the utility of customer A, we obtain

$$\begin{aligned} \frac{u(y_A)}{u(x_A)} &= \left(1 + \frac{1}{8\lambda}\lambda\right)^{1/2} \left(1 - \frac{4}{5}\lambda \frac{1}{8\lambda}\right)^{1/2} \\ &= \left(1 + \frac{1}{8}\right)^{1/2} \left(1 - \frac{4}{5}\frac{1}{8}\right)^{1/2} = \left(\frac{9}{8}\frac{9}{10}\right)^{1/2} = \left(\frac{81}{80}\right)^{1/2} > 1. \end{aligned}$$

Substituting to the equations for the utility of customer B, we obtain

$$\begin{aligned} \frac{u(y_A)}{u(x_A)} &= \left(1 - \frac{2}{3}\frac{1}{8}\right)^{1/2} \left(1 + \frac{4}{5}\frac{1}{8}\right)^{1/2} \\ &= \left(1 + \frac{1}{10}\right)^{1/2} \left(1 - \frac{4}{5}\frac{1}{10}\right)^{1/2} = \left(\frac{110}{100}\frac{92}{100}\right)^{1/2} > 1, \end{aligned}$$

Extra space for question part (c) of Q3.

4. (25) Consider an exchange economy with I consumers, and L goods. The initial endowments are $\omega_i \in \mathbb{R}^L, \omega_{il} \gg 0$ for each i and l, and for each i, the utilitiy function u_i of agent i is strongly monotone.

   (a) (5) State the definition of a Walrasian equilibrium for this economy *Solutions:* A Walrasian equilibrium is a allocation $x = (x_1, ..., x_I)$ and a vector of non-negative prices p such that

   - (*consumer's optimization*) for each i, for any $y \in X^L$, either $u_i(y) \leq u_i(x_i)$ or $p \cdot y > p \cdot x_i$,
   - (*market clearing*) x is feasible:

$$\sum_i x_i \leq \sum_i \omega_i$$

(b) (5) State the definition of the core. *Solutions:* A feasible allocation x (i.e., an allocation such that $\sum_i x_i \leq \sum_i \omega_i$) can be blocked by coalition $S \subseteq I$ if there is an "$S-$allocation" $(x_i')_{i \in S} \in \prod_{i \in S} X_i$ that (a) satisfies (weak) S- aggregate feasibility:

$$\sum_{i \in S} x_i' \leq \sum_{i \in S} \omega_i,$$

and such that (b) for each $i \in S$, $u_i(x_i') > u_i(x_i)$. A core is a set of feasible allocations in the whole economy that cannot be blocked by any coalition.

(c) (15) Prove (using only the definitions) that any Walrasian equilibrium allocation is in the core. *Solutions:* Suppose that (x, p) is a Walrasian equilibrium. Because of strongly monotone utilities, it must be that the prices are strictly positive. Suppose that there is an "$S-$allocation" $(x'_i)_{i \in S} \in \prod_{i \in S} X_i$ that (a) satisfies (weak) S- aggregate feasibility:

$$\sum_{i \in S} x'_i \leq \sum_{i \in S} \omega_i,$$

and such that $u_i(x'_i) > u_i(x_i)$. Because of the latter, it must be that none of the bundles x'_i is available at prices p:

$$p \cdot x'_i > p \cdot \omega_i.$$

By summing over $i \in S$, we get

$$\sum_{i \in S} p \cdot x'_i > \sum_{i \in S} p \cdot \omega_i.$$

But, because the prices are positive, this contradicts S-feasibility.

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SPACE FOR ROUGH WORK ONLY. WORK HERE WILL NOT BE GRADED