COMPARATIVE STATICS

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Please let me know about any typos, mistakes, unclear or ambiguous statements that you find.

1. Comparative statics

In many economic problems, we are interested in how the maximizers of a certain function change with some parameters of the function. Let X be a space of choices, and T be a space of parameters. Let $f : X \times T \to R$ be a function. We want to know how the utility maximizers

$$x^*(t) = \arg\max_x f(x,t) \tag{1.1}$$

change with parameters t. If the maximizer is not unique, then $x^*(t)$ is a set.

Examples include: utility maximization given some parameter, game theory (x is the best response, and t is the action of the opponent.)

Example 1. Neoclassical firm chooses capital k and labor l to maximize profits and to produce target output y:

$$\max_{k,l} py - rk - wl \text{ st. } F(k,l) = y.$$

How does the amount of l depend on y? It seems natural to expect that l increases with y - what assumptions on F are necessary to show it?

Another cute example comes from (Quah and Strulovici, 09)

Example 2. *Optimal stopping problem.* Suppose that $u(t)$ is a flow of payoffs in time t from certain activity. The agent discounts future payoff with discount factor δ. The agent may decide to terminate the activity in period x; in such case, his payoff is equal to

$$V_\delta(x) = \int_0^x e^{\delta t} u(t)\,dt.$$

The optimal stopping time is an element of a set $\arg\max_x V_\delta(x)$. How does the optimal stopping change with the agent's patience? The intuition says that more patient agents wait longer. Is it correct?

2. Implicit Function Theorem

If everything is nice and differentiable, and the maximizer is unique we can analyze the derivative $\frac{dx^*(t)}{dt}$. Assume that $X, T \subseteq R$ and that u is twice continuously differentiable, and $f_{xx} < 0$. Then, $f_x(x,t) = 0$ characterizes the x-maximizer of u. In particular,

$$f_x(x^*(t), t) = 0 \text{ and } f_{xx}(x^*(t), t) < 0.$$

(The second condition is a strong version of the second order condition, and sufficient condition for the maximum.) By differentiating the first identity, we get

$$f_{xx}(x^*(t), t) \frac{dx^*(t)}{dt} + f_{xt}(x^*(t), t) = 0,$$

which implies that

$$\frac{dx^*(t)}{dt} = -\frac{f_{xt}(x^*(t), t)}{f_{xx}(x^*(t), t)}.$$

Thus, the change of the maximizer depends on the second derivatives of function f.

Problems:

- We need lots of assumptions: differentiability, uniqueness, the existence of inverse of f_{xx} (with more than one variables, this means that the matrix
- There are some context in which some of these assumptions are strong.
- Sometimes, they are impossible (in games, where f is an endogenous object, like best response).

3. Univariate case

Things are not always nice and differentiable, and the optimal solutions are not always unique. In order to deal with such situations, we need a better, more robust theory of comparative statics.

We develop this theory in two stages. First, we assume that X and T are subsets of the real line R. This is the most familiar case, and arguably, having most decision-theoretic applications.

Because the maximizers (1.1) might not be unique, we need to define what we mean that one set of maximizers is larger than the other. We use so-called *strong set order* (Topkis). Take any two subsets $A, B \subseteq X$: we say that A is (weakly) dominated by B in the strong set order sense, write $A \leq_S B$, if for each $a \in A$, and $b \in B$, $\min(a,b) \in A$ and $\max(a,b) \in B$.

Example 3. On real line $[3,5] \leq_S [6,7]$, $[3,5] \leq_S [4,7]$, but not $[3,5] \leq_S [2,7]$ nor $\{3,5\} \leq_S \{4,7\}$.

Say that correspondence $x^* : R \rightrightarrows R$ is (weakly) increasing if for any $t < t'$, $x^*(t) \leq_S x^*(t')$.

3.1. **Single crossing condition (Milgrom and Shannon).** Say that family of functions $f(.,t)$ is ordered by the *single crossing condition*, if

$$f(x',t) - f(x,t) \geq 0 \text{ implies } f(x',t') - f(x,t') \geq 0, \text{ and}$$
$$f(x',t) - f(x,t) > 0 \text{ implies } f(x',t') - f(x,t') > 0$$
$$\text{for all } t < t' \text{ and } x < x'.$$

Theorem 1. *Let $x^*(t)$ be a solution to (1.1). If family $f(.,t)$ is ordered by the single crossing condition, then, $x^*(t)$ is increasing.*

Proof. Take any $x \in x^*(t)$ and $x' \in x^*(t')$ for some $t < t'$ and suppose that $x > x'$. We will show that $x' \in x^*(t)$. Indeed, if not then $f(x',t) < f(x,t)$. Because $x' < x$, the single crossing condition implies that $f(x',t') < f(x,t')$, which contradicts the fact that $x' \in x^*(t')$.

Similarly, we show that for any $x \in x^*(t)$ and $x' \in x^*(t')$ such that $t < t'$ and $x > x'$, it must be that $x \in x^*(t')$. Because $x \in x^*(t)$, it must be that $f(x',t) \leq f(x,t)$. Because $x' < x$ and by the single crossing condition, $f(x',t') \leq f(x,t')$, which implies that $x \in x^*(t')$. □

One shows that the single-crossing property is necessary for the monotone comparative statics in the following (very strong!) sense. For any set of constraints $Y \subseteq R$, consider the constrained problem

$$x^*_Y(t) = \arg\max_{x \in Y} f(x,t).$$

Theorem 2. *The following statements are equivalent:*

(1) *Family $f(.,t)$ is ordered by single crossing condition.*
(2) *For each set $Y \subseteq R$, each $t < t'$,*

$$x_Y^*(t) \leq_S x_Y^*(t').$$

Proof. Direction (1) to (2) follows from the proof of Theorem 1 without any modifications. (Notice that theorem 1 asserted somehow weaker claim).

Direction (2) to (1). We prove by contradiction. Suppose that (2) holds and (1) does not hold. Then, there exist $x < x'$ and $t < t'$ that violates one of the two conditions from the definition of the SCC. Assume that the first condition is violated (the proof in the case of the second condition is similar). Then,

$$f(x',t) \geq f(x,t) \text{ and } f(x',t') < f(x,t').$$

Let $Y = \{x, x'\}$. Then, the first inequality implies that

$$x' \in x_Y^*(t),$$

and the second inequality implies that

$$x' \notin x_Y^*(t') = \{x\}.$$

Thus,

$$\text{not } x_Y^*(t) \leq_S x_Y^*(t').$$

Contradiction. □

(See also a similar result below about increasing differences).

3.2. **Increasing differences.**

Function $f : R^2 \to R$ has *increasing differences*, if for each $x' > x$, the difference

$$f(x',t) - f(x,t)$$

is (weakly) increasing in t. Function f has strictly increasing differences if the difference is strictly increasing.

Consider maximization problem

$$x_Y^*(t) = \max_{x \in Y} f(x,t) + g(x). \tag{3.1}$$

The next result shows that increasing differences is a sufficient and necessary condition for the set of maximizers to be increasing.

Theorem 3. *(RCS) The following statements are equivalent:*

(1) *Function f has increasing differences.*
(2) *The solution $x_Y^*(t)$ to maximization problem (3.1) is (weakly) increasing in t for all functions g and sets Y.*

Proof. The first direction follows from Theorem 1. (Check that if f has increasing differences, then also $f+g$, which implies that $f(.,t)+g(.)$ is ordered by the single crossing-condition). For the other direction, suppose that there exists $t<t'$ and $x<x'$ such that

$$f(x',t)-f(x,t)>f(x',t')-f(x,t').$$

Let $Y=\{x,x'\}$ and find function g such that $g(x)=0$ and $g(x')=f(x,t)-f(x',t)$. Then,

$$x'\in x_Y^*(t)=\arg\max_{y\in Y} f(y,t)+g(y)$$

(because $f(x,t)+g(x)=f(x',t)+g(x')$), but

$$x'\notin x_Y^*(t')=\arg\max_{y\in Y} f(y,t')+g(y)$$

(because $f(x,t')+g(x)>f(x',t')+g(x')$). This contradicts (2). □

Remark 1. (Quah and Strulovici, 09) point that the second part of the above result heavily depends on the fact that we allow any set Y in the optimization problem (3.1). If, instead, we restrict Y to be intervals, then the sufficient and necessary conditions are weaker than increasing differences.

The next result provides a simple way to check condition that ensures increasing differences in the differentiable case.

Lemma 1. *If function f is twice differentiable, then family $f(.,t)$ has increasing differences iff $f_{xt}\geq 0$.*

Proof. For each $x < x'$, each t, define

$$g_{x,x'}(t) = f(x', t) - f(x, t) = \int_x^{x'} f_x(u, t)\, du.$$

Increasing differences implies that function $g_{x,x'}$ must be (weakly) increasing for each $x < x'$ and t. In the differentiable case, this means that $g'_{x,x'}(t) \geq 0$ for each t, or

$$\int_x^{x'} f_{xt}(u, t)\, du \geq 0$$

for each t. Because the above holds for each $x < x'$, it must be that $f_{xt} \geq 0$. □

3.3. **Some applications.** In another example, we show a simple proof that monopolist always produces less than socially optimal.

Example 4. Suppose that the one-to-one demand curve for a good produced by a monopolist is $x(p)$ so that the consumer surplus is equal to

$$CS(p) = \int_p^{\infty} x(r)\, dr.$$

(Precisely, that comes from a model in which the consumer chooses demand when maximizing utility with quasi-linear preferences over money.) Then, the consumer surplus function is non-decreasing. Let $p(x)$ be the inverse demand function, i.e. $p(x(p)) = p$. The monopolist's profits are equal to

$$\pi(x) = xp(x) - c(x),$$

where $c(x)$ is the cost of producing quantity x.The maximization problem for the monopolist is

$$\max_x \pi(x).$$

The maximization problem for the society is

$$\max_x \pi(x) + CS(p(x)).$$

For each x and $t \in [0, 1]$, define

$$f(x, t) = \pi(x) + tCS(p(x)).$$

We can check that function f satisfies single-crossing condition (Check !). Hence, $x(0) < x(1)$ and the monopolist produces less then it is socially optimal!

The next problem is savings-consumption problem

Example 5. An agent chooses consumption in two periods, $c_1 = w - s$ and $c_2 = s$, where s are the first-period savings. She has a discounted and separable utility over two period consumption

$$u_1(w-s) + \beta u_2(s),$$

where we assume that functions u_i are weakly increasing. We emphasize: monotonicity of u is the only assumption that we are going to use!

Consider function

$$f(s;\beta) = u_1(w-s) + \beta u_2(s).$$

In an exercise, you are supposed to show that the function has increasing differences. By Theorem 1, this implies that the savings increase in the discount factor (or, more precisely, in how much the agent weighs the future). This is good and expected, but notice that we managed to obtain this result without any differentiability or convexity assumptions on the utility function!

4. Multivariate case

In many applications, the choice variable is multidimensional. For example, consider the profit maximization problem

$$\max_{k,l} pf(k,l,t) - r(k) - w(l),$$

where the cost of capital and labor are some, possibly, non-linear, functions of the inputs. We would like to know how the optimal level of inputs changes with some parameter t.

4.1. **Order and Lattices.**

This and the subsequent sections develop the theory of comparative statics for more general spaces X and T.

Recall that set X is *partially ordered* by $\leq$ if for each $x, y, z \in X$ (a) $x \leq x$, (b) if $x \leq y$ and $y \leq x$, then $x = y$, and (c) if $x \leq y$ and $y \leq z$, then $x \leq z$. We say that the order is *total*, if for each x,y$\in X$, either $x \leq y$ or $y \leq x$.

Definition 1. Set X with partial order $\leq$ is a *lattice,* if for each $x, y \in X$, there exist

- the unique element z such that $x \leq z, y \leq z$, and for each z' so that $x \leq z', y \leq z'$, it must be that $z \leq z'$. We say that z is the *join* (or "least upper bound", or "maximum) of x and y, and write $z = x \vee y$,
- the unique element z such that $z \leq x, z \leq y$, and for each z' so that $z' \leq x, z' \leq y$, it must be that $z' \leq z$. We say that z is the *meet* (or "greatest lower bound", or "minimum") of x and y, and write $z = x \wedge y$.

Definition 2. Suppose that $(X, \leq)$ is a lattice. For all subsets $A, B \subseteq X$, write $A \leq_S B$ iff for each $x \in A, y \in B$, $x \wedge y \in A$, and $x \vee y \in B$.

Example 6. Let $X_1, X_2 \subseteq R$ be two closed intervals. Let $X = X_1 \times X_2 \subseteq R^2$ be a subset of R^2 that is equipped with the partial order of coordinatewise vector comparison, i.e., for each $x = (x_1, x_2), y = (y_1, y_2) \in X$, we have $x \leq_X y$ if and only if $x_n \leq x'_n$ for each n. We will show that X is a lattice.
Indeed, for each $x = (x_1, x_2), y = (y_1, y_2) \in X$,

- let $x \vee y = (\max(x_1, y_1), \max(x_2, y_2))$ and notice that (a) $x \vee y \geq_X x, y$ and (b) for each $z \geq_X x, y$, it must be that $z_i \geq x_i, y_i$, which implies that $z_i \geq \max(x_i, y_i)$, and $z \geq_X x \vee y$. Thus, $x \vee y$ is a well-defined and unique join of the lattice,
- Let $x \wedge y = (\min(x_1, y_1), \min(x_2, y_2))$and notice that (a) $x \wedge y \leq_X x, y$ and (b) for each $z \leq_X x, y$, it must be that $z_i \leq x_i, y_i$, which implies that $z_i \leq \min(x_i, y_i)$, and $z \leq_X x \wedge y$,Thus, $x \wedge y$ is a well-defined and unique meet of the lattice.

See Fig. 4.1. We refer to $(X_1 \times X_2, \leq_{X_1 \times X_2})$ as a product lattice.

Exercise 1. We extend the above example to multiple dimensions. Suppose that $X_1, ..., X_n \subseteq R$ are closed intervals. Show that the product set $X^* = X_1 \times ...X_n$ is a lattice under the partial order of coordinatewise vector comparison (i.e., for each $x, y \in X^*$, we have $x \leq y$ if and only if $x_i \leq y_i$ for each $i \leq n$.). What is the joint and the meet in this lattice? Explain what the set order means.
This example can be generalized even further. Even more generally, suppose that

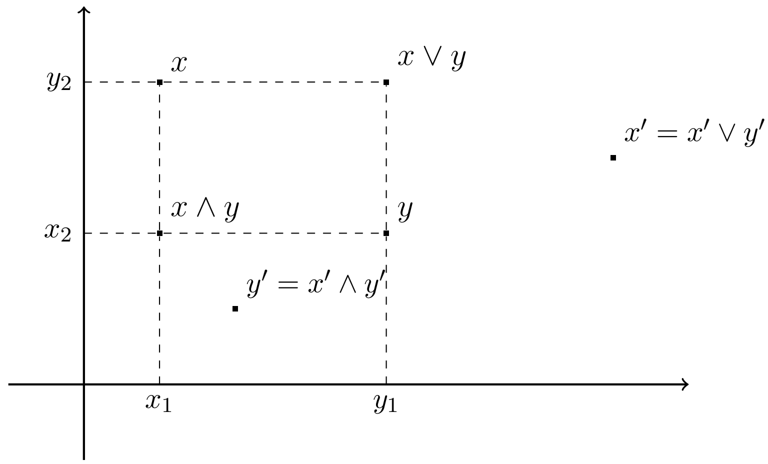

FIGURE 4.1. Two-dimensional Euclidean lattice. Here, (a) neither $x \leq y$ nor $y \leq x$, and (b) $y' \leq x'$.

$(X_1, \leq_1), ..., (X_n, \leq_n)$ is a collection of lattices. Let $X = X_1 \times ... \times X_n$ be a Cartesian product of the lattice spaces. Define a binary relation $\leq_X$ on X: for each $x = (x_1, ..., x_n), y = (y_1, ..., y_n) \in X$, let $x \leq y$ if and only if $x_i \leq_i y_i$ for each i. Show that $(X, \leq)$ is a lattice. (Is it necessary that spaces $(X_i, \leq_i)$ are lattices? Would it be enough to assume that they are partial orders?) We refer to $(X, \leq)$ as a product lattice.

4.2. **Supermodularity.** Suppose that $(X, \leq)$ is a lattice. Function $f : X \to R$ is *supermodular* if for all $x, y \in X$,

$$f(x) + f(y) \leq f(x \wedge y) + f(x \vee y).$$

It is *strictly supermodular*, if the inequality is strict for all x, y that cannot be compared (i.e., so that neither $x \leq y$ nor $y \leq x$).

Intuition:

- the effect of the increase in one variable is larger if the other variable also increases,
- one way to think about supermodularity is that it gives a precise mathematical definition for complementarities.

Example 7. Show that function $f(x, y) = axy$ for constant $a \geq 0$ is supermodular.

Exercise 2. Suppose that $X_1, ..., X_n \subseteq R$ are open intervals and assume that $X = X_1 \times ...X_n$ is the product lattice from Example 1. Show that function f is (strictly) supermodular if and only if for each $x_0 \in X^*$, for each $k, l \leq n$,

$$\frac{\delta^2 f}{\delta x_k \delta x_l}(x_0) \geq (>)\, 0.$$

Definition 3. Suppose that $(X, \leq_X)$ and $(T, \leq_T)$ are partially ordered sets. Say that function $f : X \times T \to R$ has *increasing differences*, if for each $x \leq x'$, and $t \leq t'$,

$$f(x', t) - f(x, t) \leq f(x', t') - f(x, t').$$

The next exercise shows that increasing differences assumption is a weaker version of supermodularity (for example, it does not require sets to be lattices)

Lemma 2. *Suppose that $(X, \leq_X)$ and $(T, \leq_T)$ are lattices, $(X \times T, \leq^*)$ is the product lattice, and $f : X \times T \to R$ is supermodular. Then, f has increasing differences.*

Proof. Exercise. □

Recall that for any function $f : X \times T \to R$,

$$x^*(t) = \arg\max_x f(x, t).$$

Theorem 4. *(Topkis) Suppose that $(X, \leq_X)$ is a lattice, $(T, \leq_T)$ is a partially ordered set, and $f : X \times T \to R$ is a function with increasing differences and such that $f(., t)$ is supermodular for each t. Then, for each $t \leq t'$, $x^*(t)$ is a lattice (i.e., for each $x, x' \in x^*(t)$, $x \wedge x' \in x^*(t)$ and $x \vee x' \in x^*(t)$, and*

$$x^*(t) \leq_{Strong} x^*(t').$$

Proof. We start with the first claim. Suppose that $x, x' \in x^*(t)$ and $x \wedge x' \notin x^*(t)$. Then,

$$f(x \vee x') \geq f(x) + f(x') - f(x \wedge x') > f(x),$$

which contradicts the fact that $x \in x^*(t)$.

Next, we show the second claim. Take any $x \in x^*(t)$ and $x' \in x^*(t')$. By supermodularity,

$$0 \leq f(x, t) - f(x \wedge x', t) \leq f(x \vee x', t) - f(x', t).$$

By increasing differences,

$$f(x \vee x', t) - f(x', t) \leq f(x \vee x', t') - f(x', t').$$

Because $x' \in x^*(t')$, the two inequalities imply that $f(x \vee x', t') - f(x', t') \geq 0$ and $x \vee x' \in x^*(t')$. In a similar way, we show that $x \wedge x' \in x^*(t)$. □

4.3. **Example.** A monopolist chooses quantity q and quality e of its product. The profit function is equal to

$$q(A(e) + P(q)) - W(e) - cC(q).$$

We assume that function A is increasing in e and it can be interpreted as a measure of the shift in the demand induced by the quality level e. $W(e)$ is the cost of quality e. The cost of producing quantity q is equal to $cC(q)$, where C is strictly increasing and we use parameter $c > 0$ to analyze the impact of changing costs on the monopolist's decisions.

Notice that in order to guarantee that the monopolist problem has a unique solution, we need to make lots of assumptions about the shape (convexity) of functions W and C. Nevertheless, we can say quite a lot about the monopoly problem without making any of these assumptions. In particular, we will show, without any further assumptions, that the optimal quantity, and the optimal choice of quality are decreasing with the marginal cost c.

To see that, let $\theta = -c$, and define function

$$f(q, e; \theta) = q(A(e) + P(q)) - W(e) + \theta C(q).$$

Consider a standard product partial ordering of couples: $(q, e) \leq^* (q', e')$ if and only if $q \leq q'$ *and* $e \leq e'$. We will show below that function $f(.,.;\theta)$ is supermodular for each θ and it is ordered by increasing differences. (As it will be clear soon, the reason we change the variable from c to $\theta = -c$ is to ensure the increasing differences).

Before that, notice that this observation, together with Theorem 4 proves our comparative statics observation. Indeed, let

$$D^*(\theta) = \arg\max_{(q,e)} q(A(e) + P(q)) - W(e) + \theta C(q)$$

be the set of optimal decisions over quantity q and quality e . Then, Theorem 4 implies that $D^{*}(\theta)$ increases in the strong set order in θ. Because of the definition of θ, it means that $Deach^{*}(\theta)$ decreases in c. (Be careful that you understand what exactly is the definition of the strong set order used here.)

Next, we will show that function $f(.,.;\theta)$ is supermodular. We need to show that for each q,e and q',e',

$$f(q,e;\theta)+f(q',e';\theta)\leq f(\min(q,q'),\min(e,e');\theta)+f(\max(q,q'),\max(e,e');\theta).$$

Indeed, let's compute the difference between the right- and the left-hand side. After cancelling bunch of terms, we get

$$\max(q,q')A(\max(e,e'))+\min(q,q')A(\min(e,e'))-qA(e)-q'A(e').$$

Suppose w.l.o.g. that $e<e'$. Then, the above is equal to

$$(\max(q,q')-q')A(e')+(\min(q,q')-q)A(e)$$
$$(\text{if } q<q')=0,$$
$$(\text{if } q\geq q')=(q-q')(A(e')-A(e))\geq 0,$$

where the last inequality comes from the fact that A is increasing.

Finally, we check that family $f(.,.;\theta)$ is ordered by increasing differences. Notice that for each $q\leq q'$ and $e\leq e'$,

$$f(q',e';\theta)-f(q,e;\theta)=\theta(C(q')-C(q)).$$

Because C is increasing, the term in the bracket is positive, and the above expression is increasing in θ.

4.4. **Application: Global Le Chatelier Principle.** The Le Chatelier principle says that the short-run effects of the wage increase on the labor demand are smaller than the long-run effects. The principle applies only locally, to small changes in wage. Given some stronger (supermodularity) assumptions on the production function, we can show a global version of the Le Chatelier principle. We follow Milgrom-Roberts (96).

Suppose that $(X_1, \leq_X)$ and $(X_2, \leq_Y)$ are compact lattices, $X = X_1 \times X_2$ is a product lattice (with partial order defined as in Example 1), and $(T, \leq_T)$ is a partially ordered set. Let $f X \times T \to R$ be a continuous function. Consider two optimization problems:

$$\text{Problem A: } x_1^* (x_2, t) = \arg\max_{x_1 \in X} f(x_1, x_2, t),$$

$$\text{Problem B: } x^* (t) = \arg\max_{x \in X} f(x_1, x_2, t).$$

In problem A, we maximize the value of function f by choosing optimal x_1 and keeping x_2. In Problem B,we allow to vary both x_1 and x_2.

Lemma 3. *If function f is continuous and supermodular for each t, then there exists a unique element*

$$\left(x_1^{sup}(t), x_2^{sup}(t)\right) = \bigvee_{x \in x^*(t)} x \in x^*(t).$$

Proof. Notice that $x^*(t)$ is compact lattice, which implies that it contains the largest element. □

Theorem 5. *Suppose that continuous function $f : X \times Y \times T \to R$ is supermodular in (x, y) for each t and it has increasing differences in (x, y) and t. is continuous and supermodular (given the partial order on lattice R^3) . Then, for each $t < t'$,*

$$x_1^*\left(x_2^{sup}(t), t\right) \leq_S x_1^*\left(x_2^{sup}(t), t'\right) \leq_S x_1^*\left(x_2^{sup}(t'), t'\right).$$

Here,

$$x_1^*\left(x_2^s(\theta_0), \theta_1\right)$$

is the set of optimal choices of x_1 in the problem A given θ_1 if the value of the constrained variable is set at the level that is the largest possible among optimal choices given the parameter θ_0.

The first inequality corresponds to the short-run effect of the change in t on x: if t increases, more x_1 is optimal. The second inequality corresponds to the long-run effect. If the optimization criterion is supermodular, then the short-run effect is smaller than the long-run effect.

Proof. By Theorem 4,

$$x^*(t) \leq_S x^*(t'),$$

which implies that

$$x_2^{sup}(t) \leq x_2^{sup}(t').$$

Because f is supermodular in (x_1, x_2), it is also supermodular in x_1 when the value of x_2 is fixed. Also, it has increasing differences in x_1 and t for fixed value of x_2. Thus, we can apply Theorem 4 to the constrained problem, where the value x_2 is set at level $x_2^{sup}(t)$. We obtain

$$x_1^*(x_2^{sup}(t), t) \leq_S x_1^*(x_2^{sup}(t), t').$$

Because f is supermodular in (x_1, x_2) for fixed t, it is also supermodular in x_1 when the values of x_2 and t are fixed. Thus, we can apply Theorem 4 to a problem, where the parameter is fixed at t' and we treat x_2 as a new parameter, the value of which changes from $x_2^{sup}(t)$ to $x_2^{sup}(t')$. An application of the Theorem shows that

$$x_1^*(x_2^{sup}(t), t') \leq_S x_1^*(x_2^{sup}(t'), t').$$

The result follows. □

Corollary 1. ***(Global Le Chatelier Principle)*** *Consider a two-input production function $g(k,l)$ and suppose that either g (or $-g$) is supermodular or (in the latter case, we say that g is submodular). Let $l^*(k,w,r)$ be the short-run optimal level of labor given the level of capital and factor prices, and let $k^{sup}(w,r)$ be the long-run largest optimal level of capital. Then, for each $w > w'$, (or, $w < w'$)*

$$l^*(k^{sup}(w,r), w, r) \leq l^*(k^{sup}(w,r), w', r) \leq l^*(k^{sup}(w',r), w', r).$$

Proof. If g is supermodular, then let $f(k,l,w) = g(k,l) - wl - rk$. If g is submodular, then let $f(k,x,w) = g(k,-x) + wx - rk$. Apply Theorem 5. □