MICROECONOMIC THEORY QUESTIONS
MIDTERM

MARCIN PĘSKI

Each question has the same value. You need to provide arguments for each answer. If you cannot solve one part of the problem, don't give up and try to solve the next one. If the question explicitly asks you to prove a result from the class, you must carefully describe the proof. Otherwise, you may use any result from the class given that you clearly state the assumptions, thesis and verify that the assumptions hold in your application. You have 110 minutes. Good luck!

1. Consumer and Firm Theory

(1) Answer the following questions.

(a) Show that Slutsky matrix is symmetric and negative semi-definite.

Slutsky matrix is the derivative matrix of the Hicksian demand, $S = D_p h$. By the Shepherd's Lemma,

$$h = D_p e\left(p; u\right).$$

It follows that

$$S = D_p^2 e\left(p; u\right).$$

The symmetry follows. The negativeness comes from the fact that $e\left(p, u\right)$ is concave in p.

(b) State and prove the Law of Compensated Demand.

Because S is negative semi-definite, it follows that for each good l, its diagonal elements are non-positive. Thus,

$$\frac{\delta h_l}{\delta p_l} \leq 0.$$

(2) Suppose that $\mathcal{X} = \{(x_1, ..., x_L) : x_l \geq 0\}$ and $\mathcal{Y} = \{(y_1, ..., y_K) : y_k \geq 0\}$ are the consumption spaces over two different baskets of goods. Consumer's utility over consumption bundles $(x, y) \in \mathcal{X} \times \mathcal{Y}$ is equal to

$$u(x, y) = h(x) + g(y),$$

where h and g are functions representing continuous, strictly convex, and strictly increasing preferences over baskets, respectively, $\mathcal{X}$ and $\mathcal{Y}$.

Let

$$v_h(p_X, w_X) := \max_{x \in \mathcal{X}} h(x) \text{ st. } p_X \cdot x \leq w_X,$$
$$v_g(p_Y, w_Y) := \max_{y \in \mathcal{Y}} g(y) \text{ st. } p_Y \cdot y \leq w_Y,$$

be the indirect utility functions associated with each of the functions h and g. Explain that the indirect utility function from the original problem is equal to

$$v(p_X, p_Y, w) := \max_{w_X} v_h(p_X, w_X) + v_g(p_Y, w - w_X).$$

- Fix p and w and find the unique solution $(x(p, w), y(p, w))$ to the consumer problem. Let

$$w_X^* = p_X \cdot x(p, w).$$

Because $p_X \cdot x(p, w) \leq w_X^*$, bundle $x(p, w)$ is available for the consumer with preferences $h(x)$ at prices p_x and wealth w_X^*. This implies that

$$h(x(p, w)) \leq \max_{x : p_X \cdot x \leq w_X^*} h(x) = v(p_X, w_X^*).$$

Similarily, $p_y \cdot y(p, w) \leq w - p_X \cdot x(p, w) = w - w_X^*$,

$$g(y(p, w)) \leq v_g(p_Y, w - w_X^*).$$

It follows that

$$\begin{aligned} v(p,w) &= u(x(p,w), y(p,w)) \\ &= h(x(p,w)) + g(y(p,w)) \\ &\leq v(p_X, w_X^*) + v_g(p_Y, w - w_X^*) \\ &\leq \max_{w_X} v_h(p_X, w_X) + v_g(p_Y, w - w_X). \end{aligned}$$

- On the other hand, let $x_h(p_X, w_X)$ be the solution to the problem $\max_{x\in\mathcal{X}} h(x)$ st. $p_X \cdot x \leq w_X$. Analoguously, let $y_g(p_Y, w_Y)$ be a solution to $\max_{y\in\mathcal{Y}} g(y)$ st. $p_Y \cdot y \leq w_y$. Then, for each w_X,

$$p_X \cdot x_h(p_X, w_X) + p_Y \cdot y_g(p_Y, w - w_X) \leq w_X + w - w_X = w.$$

This implies that for each w_X, bundle $(x_h(p_X, w_X), y_g(p_Y, w_Y))$ is available as a choice for the consumer problem. It follows that for each w_X,

$$\begin{aligned} v_h(p_X, w_X) + v_g(p_Y, w - w_X) &= h(x_h(p_X, w_X)) + g(y_g(p_Y, w_Y)) \\ &= u(x_h(p_X, w_X), y_g(p_Y, w_Y)) \\ &\leq \max_{x,y: p_X\cdot x + p_Y \cdot y \leq w} u(x,y) \\ &= v(p,w). \end{aligned}$$

Because the above inequality holds for each w_X, we get

$$\max_{w_X} v_h(p_X, w_X) + v_g(p_Y, w - w_X) \leq v(p,w).$$

The result follows.

(3) Answer the following questions.

(a) State the definition of a lattice.

Set X with partial order $\leq$ is a *lattice,* if for each $x, y \in X$, there exist

- the unique element z such that $x \leq z, y \leq z$, and for each z' so that $x \leq z', y \leq z'$, it must be that $z \leq z'$. We say that z is the *joint* (or "maximum") of x and y, and write $z = x \vee y$,
- the unique element z such that $z \leq x, z \leq y$, and for each z' so that $z' \leq x, z' \leq y$, it must be that $z' \leq z$. We say that z is the *meet* (or, "minimum") of x and y, and write $z = x \wedge y$.

(b) Let $X = R^2_+$ be a vector space with the vector partial order (i.e., for each $x = (x_1, x_2), y \in X$, $x \leq_X y$ if and only if $x_i \leq y_i$ for each $i = 1, 2$.) Is X a lattice? Why?

For each $x, y \in X$, define vectors $x \wedge y$ and $x \vee y$ so that for each $i = 1, 2$,

$$(x \wedge y)_i = \min (x_i, y_i) \text{ and } x \vee y = \max (x_i, y_i) .$$

Clearly $x \wedge y, x \vee y \in X$

We will check that the above operations are properly defined lattice operations. Notice that for each $z \geq x, y$, we have $z_i \geq \max (x_i, y_i)$ for each i, which implies that $z \geq x \vee y$. A similar argument works demonstrates that for each $z \leq x, y$, we have $z \leq x \wedge y$.

(c) Let $h(x) = \min (x_1, x_2)$ and $g(x) = \max (x_1, x_2)$. Which of the functions h and/or g are supermodular?

We will show that f is supermodular. It is enough to show that for each $t < t'$, the difference

$$f(x) := h(x, t') - h(x, t) = \min (x, t') - \min (x, t) .$$

is increasing in x. However, notice that

$$f(x) = \begin{cases} 0, & \text{if } x \leq t \\ x - t, & \text{if } x \in [t, t'], \\ t' - t, & \text{if } x \geq t', \end{cases}$$

and $f(.)$ is increasing function.

We will show that g is not supermodular. Indeed, notice that

$$g(1,2) + g(2,1) = 4 \geq 3 = g(1,1) + g(2,2).$$

(4) Larry Boy got picked up at 4am raid from his cottage in the woods surrounding the Lakahoma Lake. After six hours of sitting alone in the cell, officer Norck enters and explains that the police and the prosecution know but they still cannot prove that on the order of Tall Luiggi, Larry Boy murdered Bobby Romano and sank his body in the Lakahoma Lake. They are still looking for the body and once they find it, Larry Boy gets death sentence for the murder. Larry Boy worries that the police will find the body if Bobby Romano's shoes were to fall off. The job was rushed and Larry Boy is not exactly sure whether the concrete was done properly. If the shoes stay on, and the police doesn't find the body, Larry Boy will go free for the lack of evidence.
Larry Boy has another choice: He can talk to the police about Tall Luiggi's organization and repeat everything in the court. In such a case, officer Norck promises to cancel the search for Bobby Romano's body (which means no murder charge for Larry) and try to find a place for Larry Boy in the witness protection program. Larry Boy is aware that testifying for the police does not guarantee safety: Tall Luiggi's arms reach far and in the past, approximately 30% of all star witnesses were killed before they joined the witness protection program.

(a) Describe the Larry Boy's two choices (i.e., Staying Silent and Talking to Officer Norck) using Anscombe-Aumann acts over state space $S = \{CF, CS\}$, where CF means "concrete shoes fall off", and CS means concrete shoes stay on" and prizes $Z = \{d, f, w\}$, where d, f, and w mean respectively, death, freedom, and life in the witness protection program.

$$f_{SS}(CS) = f, f_{SS}(CF) = d,$$
$$f_T(CS) = f_T(CF) = d^{0.3}w^{0.7}.$$

(b) Larry Boy is very tired with the constant stress of the criminal life. He would strictly prefer a Calm Retirement in the witness protection program (i.e., an act that delivers w with certainty) to Freedom (an act that delivers f in each state) and the work for Tall Luiggi. In fact, he is indifferent between Freedom and a lottery that assigns equal probability 0.4 to death and 0.6 to the witness protection program. Suppose that Larry's preferences over acts are rational. Explain, that, if Larry's preferences were to satisfy the Independence axiom, Larry would prefer Calm Retirement to Certain Death (an act that delivers d with certainty).

Let

$$\begin{aligned} f_{CM}(CF) &= f_{CM}(CS) = w, \\ f_F(CF) &= f_F(CS) = f \\ f_l(CF) &= f_l(CS) = d^{0.4}w^{0.6}, \\ f_{CD}(CF) &= f_{CD}(CS) = d. \end{aligned}$$

We have,

$$f_F \prec f_{CM} \text{ and } 0.4f_{CD} + 0.6f_{CM} \sim f_F.$$

Suppose that Larry Boy's preferences satisfy the Independence Axiom. And, on the contrary, suppose that $f_{CM} \preceq f_{CD}$. Then,

$$f_F \prec f_{CM} \preceq f_{CD}$$

and by the independence axiom,

$$f_F \prec 0.4f_{CD} + 0.6f_{CM}.$$

But this contradicts the second observation above.

(c) Suppose that Larry strictly prefers Staying Silent to Talking to Officer Norck. Are the choices described here together with the choices described in (b) are consistent with the State Independent Expected Utility? Suppose that

$$f_F \prec f_{CM} \text{ and } 0.4f_{CD} + 0.6f_{CM} \sim f_F.$$

If Larry Boy has SIEU preferences, then there exist utility function $u : \{d, w, f\} \to R$ and probability distribution $\pi \in \Delta\{CS, CF\}$that represents the above choices. Because $f_F \prec f_{CM}$ and $0.4f_{CM} + 0.6f_{CD} \sim f_F$, we must have

$$u(d) < u(w), \text{ and } u(f) = 0.4u(d) + 0.6u(w).$$

But then,

$$\begin{aligned} U(f_T) - U(f_{SS}) =& \pi_{CS}(0.3u(d) + 0.7u(w) - u(f)) \\ &+ (1 - \pi_{CS})(0.3u(d) + 0.7u(w) - u(d)) \\ =& \pi_{CS}(0.3u(d) + 0.7u(w) - 0.4u(d) + 0.6u(w)) \\ &+ (1 - \pi_{CS})\,0.7(u(w) - u(d)) \\ =& \pi_{CS}0.1(u(w) - u(d)) + (1 - \pi_{CS})\,0.7(u(w) - u(d)) > 0. \end{aligned}$$

But this contradicts

$$f_T \prec f_{SS}.$$

(d) Are the choices from (b) and (c) consistent with the Independence Axiom?

Yes. It is enough to show that there exists SDEU model that respects choices (b) and (c). For example, let $u_s(d) = 0$ and $u_s(w) = 0.5$ for each state s and let $u_{CS}(f) = 10$, $u_{CF}(f) = -9.4$. Let $U(f)$ denote the expected (state-dependent) utility of act f. Then,

$$U(f_{CR}) = 1, U(f_{CD}) = 0, U(f_F) = 0.6, U(f_{SS}) = 10, U(f_T) = 0.7.$$