Microeconomic Theory I
Midterm October 2014

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Each question has the same value. You need to provide arguments for each answer. If you cannot solve one part of the problem, don't give up and try to solve the next one. If the question explicitly asks you to prove a result from the class, you must carefully describe the proof. Otherwise, you may use any result from the class given that you clearly state the assumptions, thesis and verify that the assumptions hold in your application. You have 110 minutes. Good luck!

1. (30 points) In the economy, there are two goods and two firms $i = 1, 2$ with production sets

$$\begin{aligned} Y_1 &= [0,2] \times [0,3] \subseteq R^2, \\ Y_2 &= \{(y_1, y_2) : y_1, y_2 \geq 0 \text{ and } y_1 + y_2 \leq 2\}. \end{aligned}$$

   (a) (5 points) State the profit maximization problem of each firm.

   $\pi_i(p) = \max_{y \in Y_i} p \cdot y$.

(b) (20 points) Let $\pi_i(p)$ be the profit function of each firm. Show that there exists a set $Y^* \subseteq R^2$ such that if $\pi^*(p)$ is the profit function of a firm with technology Y^*, then for each p,

$$\pi^*(p) = \pi_1(p) + \pi_2(p).$$

Let $Y^* = Y_1 + Y_2 = \{y_1 + y_2 : y_i \in Y_i\}$.
We will show that $\pi^*(p) \leq \pi_1(p) + \pi_2(p)$. Let $y^* \in Y$ be an optimal choice of firm Y^* and prices p. Then, $y^* = y_1^* + y_2^*$ for some $y_i^* \in Y_i$. Hence,

$$\pi^*(p) = p \cdot y^* = p \cdot y_1^* + p \cdot y_2^* \leq \pi_1(p) + \pi_2(p).$$

We will show that $\pi^*(p) \leq \pi_1(p) + \pi_2(p)$. Let y_i^* be the optimal choices for firms Y_i. Then,

$$\pi_1(p) + \pi_2(p) = p \cdot y_1^* + p \cdot y_2^* = p \cdot (y_1^* + y_2^*) \leq \pi^*(p)$$

where the last inequality follows from the fact that $y_1^* + y_2^* \in Y^*$.

(c) (5 points) Does production vector $y = (4, 5)$ belong to set Y^*? What about $y' = (0, 5)$?

Because $Y^* = Y_1 + Y_2 = \{y_1 + y_2 : y_i \in Y_i\}$, it must be that $y_1 + y_2 \leq (2+3) + 2 = 7$ for each $(y_1, y_2) \in Y^*$. So, $(4, 5) \notin Y^*$. In the same time, if $y_1 = (0, 3) \in Y_1$, $y_2 = (0, 2) \in Y_2$, then $(0, 5) = y_1 + y_2 \in Y^*$.

2. (30 points) Let $(X, \mathcal{B}, C(.))$ be a choice structure such that $|X| \geq 3$. We assume that the family of choice situations $\mathcal{B}$ consists of all 3-element subsets of X (and it does not contain any other subsets). Define a binary relation $\lhd$on X: For any x, y,

$x \lhd y$ if and only if there exists $B \in \mathcal{B}$ such that $\{x, y\} \subseteq B$and $y \in C(B)$.

(a) (5 points) State the Weak Axiom of Revealed Preferences.

If for some $B \in \mathcal{B}$, $\{x, y\} \subseteq B$, we have $x \in C(B)$, then for any $B' \in \mathcal{B}$, if $\{x, y\} \subseteq B$ and $y \in C(B')$, then $x \in C(B')$.

(b) (10 points) From now on, suppose that the choice correspondence satisfies WARP. Show that the relation $\lhd$ is transitive. (Start with explaining what is "transitive".)

Suppose that $x \lhd y$ and $y \lhd z$. Thus, there exist $B, B' \in \mathcal{B}$ such that $\{x, y\} \subseteq B$ and $\{y, z\} \subseteq B'$ and $y \in C(B)$ and $z \in C(B')$. By WARP, if $x \in C\{x, y, z\}$, then $y \in C\{x, y, z,\}$, and if $y \in C\{x, y, z\}$, then $z \in C\{x, y, z\}$. Because $C\{x, y, z\} \neq \emptyset$, it must be that $z \in C\{x, y, z,\}$. Because $\{x, y, z,\} \in \mathcal{B}$, we get $x \lhd z$.

(c) (5 points) Explain that relation $\lhd$ rationalizes the choices.

The definition implies that if $x \in C(B)$, then $y \lhd x$ for each $y \in B$. In the same time, if $y \lhd x$ for each $y \in B$, then for each $y \in B$, there exists $B_y \supseteq \{x, y\}$ such that $x \in C(B')$. By WARP, if $y \in C(B)$, then $x \in C(B)$. Because $C(B) \neq \emptyset$, it must be that

(d) (10 points) Is relation $\lhd$ complete? If so, prove it. If not, provide a counterexample.

Relation $\lhd$ does not have to be complete. Suppose that $X = \{a, b, c\}$ and $a \in C(\{a, b, c\})$. Then, neither $b \lhd c$ nor $c \lhd b$.

3. (30 points) Answer the following questions.

   (a) (5 points) Let $X = R^N$ for $N \geq 2$. Describe a partial order on X for which X is a lattice.

(b) (5 points) State the definition of a supermodular function $f : X \to R$.

(c) (10 points) Suppose that $N = 2$ and suppose that $h(x_1)$ and $g(x_2)$ are increasing functions. Let $f(x_1, x_2) = h(x_1)\, g(x_2)$. Is f supermodular? Either prove that f is supermodular, or provide a counterexample.

We show that for $x_1 \geq y_1$ and $x_2 \leq y_2$,

$$\begin{aligned} f(x \vee y) + f(x \wedge y) - (f(x) + f(y)) &= h(x_1)\, g(y_2) + h(y_1)\, g(x_2) - h(x_1)\, g(x_2) - h(y_1)\, g(\\ &= h(x_1) - h(y_1)\, (g(y_2) - g(x_2)) \geq 0. \end{aligned}$$

(d) (10 points) Suppose that $N = 3$ and suppose that $h(x_1)$, $g(x_2)$, and $k(x_3)$ are increasing functions. Let $f(x_1, x_2, x_3) = h(x_1) g(x_2) k(x_3)$. Is f supermodular? Either prove that f is supermodular, or provide a counterexample.

f does not have to be supermodular. Let $h(x_1) = x_1, g(x_2) = x_2$, and $k(x_3) = x_3$. Then, $f(x_1, x_2, x_3) = x_1 x_2 x_3$. Because f is twice-differentiable, to check whether f is supermodular, it is enough to check whether the cross-derivatives are always positive. However,

$$\frac{\partial^2}{\partial x_1 \partial x_2} f(x_1, x_2, x_3) = x_3,$$

which implies that the cross-deriveative is negative for $x_3 < 0$.

4. (30 points) Let S be a finite state space, Z be a finite set of prizes, and let X be the space of Anscombe-Aumann acts $f : S \to \Delta Z$.
For each act f, for any two states s_0, s_1, define an act f^{s_0,s_1} obtained from f by exchanging the lotteries in states s_0 and s_1:

$$f^{s_0,s_1}(s) = \begin{cases} f(s_1), & \text{if } s = s_0, \\ f(s_0), & \text{if } s = s_1, \\ f(s), & \text{if } s \neq s_0, s_1. \end{cases}$$

Assume that an agent has continuous preference relation $\preceq$ over AA acts that satisfies the Independence axiom. Additionally, the preference relation satisfies the **State Invariance Axiom:** For each act f, any two states s_0, s_1, the agent is indifferent between acts f and f^{s_0,s_1}:

$$f \sim f^{s_0,s_1}.$$

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(a) (7 points) Explain that the preferences have State-Dependent Expected Utility (SDEU) representation. (You don't need to prove anything, just refer to appropriate theorem from the class. Carefully state the assumptions of the theorem.) Carefully describe the form of the representation.

The claim is implied by the theorem from the class and the fact that the preferences satisfy Continuity and Independence. The preferences are represented by

$$U(f) = \sum_{s,z} u_s(z) f_s(z)$$

for some functions $u_s : Z \to R$.

(b) (8 points) Let functions $u_s : Z \to R$ be as in the SDEU representation. Show that State Invariance implies that for each state s_0 and s_1, there exists a constant c^{s_0,s_1} such that for each z,

$$u_{s_0}(z) - u_{s_1}(z) = c^{s_0,s_1}.$$

In particular, the difference between the utilities of a prize z in any two states does not depend on z.

For any two prizes $z_0, z_1 \in Z$, consider any act f such that $f(s_0) = z_0$ and $f(s_1) = z_1$. Then, because $f \sim f^{s_0,s_1}$, we have

$$u_{s_0}(z_0) + u_{s_1}(z_1) = u_{s_0}(z_1) + u_{s_1}(z_0).$$

In particular, for any two prizes $z, x \in Z$,

$$u_{s_0}(z) - u_{s_1}(z) = u_{s_0}(x) - u_{s_1}(x).$$

It follows that the above difference does not depend on the prize z, x, but only on the states c_0, c_1.

(c) (8 points) Conclude that the preferences have a State-Independent Representation in which the decision maker acts as if she assigns equal probability to each state.

Given the observation from the previous point, fix state s^* and let $u = u_{s^*}$. It is easy to see that for any act f, (here n is the number of states)

$$
\begin{aligned}
U^*(f) &= \sum_{s,z} \frac{1}{n} u(z) f_s(z) \\
&= \frac{1}{n} \sum_{s,z} u_{s^*}(z) f_s(z) \\
&= \frac{1}{n} \sum_{s,z} \left(u_s(z) + c^{s,s^*} \right) f_s(z) \\
&= \frac{1}{n} \sum_{s,z} u_s(z) f_s(z) + \frac{1}{n} \sum_{s,z} c^{s,s^*} f_s(z) \\
&= \frac{1}{n} U(f) + \frac{1}{n} \sum_{s} c^{s,s^*}.
\end{aligned}
$$

Because monotonic transformation do not change the preferences, it follows that $U(f)$ represents the same preferences as $U^*(f)$. In particular, teh rpeferences have SIEU representation with $\pi_s = \frac{1}{n}$ and Bernoulli utility $u(.)$.

(d) (7 points) Is the representation unique? Carefully explain using any result from the class.

Any representation with SEIU preferences have unique probabilities, and the utility function is unique up to affine transformations.