Microeconomic Theory I
Midterm

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Each question has the same value. You need to provide arguments for each answer. If you cannot solve one part of the problem, don't give up and try to solve the next one. If the question explicitly asks you to prove a result from the class, you must carefully describe the proof. Otherwise, you may use any result from the class given that you clearly state the assumptions, thesis and verify that the assumptions hold in your application. You have 110 minutes. Good luck!

1. Answer the following questions:

   (a) Suppose that $u : R^L_+ \to R$ is a continuous, locally non-satiated utility function and strictly quasi-concave on consumption set $X = R^L_+$. Derive the Slutsky equation.

(b) Use the Slutsky equation and Shepherd's Lemma to establish the Law of the Compensated Demand.

(c) Explain that the Law of (Uncompensated) Demand holds for homothetic preferences.

(d) Let L be the number of inputs, let $w \in R^L_+$ be the vector of input prices, and let $f : R^L \to R$ be the production function of a firm. Establish the Law of Supply for the firm (using the Hotelling Lemma).

2. In the class, we listed necessary conditons for the Walrasian demand function $x(p, w)$ to be derived as a (possibly, set-) solution to the consumer's problem with monotonic, strictly convex, and continuous preferences. In these question, you will examine whether (some of the) conditions are empirically falsifiable, i.e., whether one can observe an empirical violation of them.

From now on, a *dataset* is a list of finitely many data points (p_i, w_i, x_i), where $x_i \in R_+^L, p_i \in R_{++}^L, w_i \in R_+$ for each i. Say that a dataset invalidates property P if there is no function $x : R_{++}^L \times R_+ \to R_{++}^L$ that satisfies property P and such that for each i, $x(p_i, w_i) = x_i$.

In each of the questions below, start your answer by stating the definition of the relevant property.

(a) Give an example of dataset that invalidates homogeneity of degree 0 in (p, w) or explain that such a dataset does not exist.

Let

$$\begin{aligned} p_1 &= (1,1), w_1 = 2, x_1 = (1,1), \\ p_2 &= (2,2), w_2 = 4, x_2 = (2,0), \end{aligned}$$

(b) Give an example of dataset that invalidates Walras Law or explain that such a dataset does not exist.

Let

$$p_1 = (1, 1), w_1 = 2, x_1 = (0, 0).$$

(c) Give an example of dataset that invalidates Weak Axiom of Revealed Preferences, but does not violate Walras Law, or explain that such a dataset does not exist. (Hint: It might be helpful to draw a picture.)

Let

$$\begin{aligned} p_1 &= (1,2)\,, w_1 = 5, x_1 = (1,2)\,, \\ p_2 &= (2,1)\,, w_2 = 5, x_2 = (2,1)\,. \end{aligned}$$

Then, the first bundle is available at the second budget set. Likewise, the second bundle is available at the first budget set. Contradiction with WARP.

(d) (Harder.) Assume that is known that $x(p,w)$ satisfies Walras Law, homogeneity and WARP, and it is continuously differentiable in p with a known bound on the derivative. Give an example of a dataset that invalidates the symmetry of the Slutsky matrix, or explain that such a dataset does not exist. (Hint: Why was the symmetry of the Slutksy matrix important for our theory?) To get a credit, describe an idea behind your answer as clearly as you can.

In one of the steps of constructing utility function from the demand, we wanted to construct the expenditure function. We used the following observation. We knew from the Slutsky equation that

$$D_p e(p,u) = x(p, e(p,u)).$$

This gives us a differential equation that helps us to recover the expenditure function. Let $p : [0,1] = R^L$ be a path of prices. We try to construct the expenditure function stepwise along the path, by going a distance $\frac{1}{N}$ at each step. Then, starting with initial condition $e(p(0), u_0) = e_0$, we can find

$$\begin{aligned} e^{(p,N)}\left(p\left(\frac{k}{N}\right), u\right) &= e^{(p,N)}\left(p\left(\frac{k-1}{N}\right), u\right) \\ &+ x\left(p\left(\frac{k-1}{N}\right), e^{(p,N)}\left(p\left(\frac{k-1}{N}\right), u\right)\right) \cdot \left(p\left(\frac{k}{N}\right) - p\left(\frac{k-1}{N}\right)\right). \end{aligned}$$

Notice that we can construct $e^{(p,N)}\left(p\left(\frac{k}{N}\right), u\right)$ using dataset with $N+1$ observations $\left(p\left(\frac{k}{N}\right), x\left(p\left(\frac{k}{N}\right), e^{(p,N)}\left(p\left(\frac{k}{N}\right), u_0\right)\right)\right)$.

By the Theorem from the differential equations, it must be that $e^{(p,N)}(p(1), u)$ converges as $N \to \infty$. Let $e^p(p(1), u)$ denotes the value of the limit. Using the fact that the bound on the derivatives of x is known, we can establish the speed of this convergence, so for each $\varepsilon > 0$ we can find N_ε such that

$$\left| e^{(p,N)}(p(1), u) - e^p(p(1), u) \right| \leq \varepsilon.$$

Moreover, if the Slutksy matrix is symmetric, than the limit does not depend on the path p or q , as long as $p(0) = q(0)$ and $p(1) = q(1)$. Thus, if we find a dataset such that

$$\left| e^{(p,N)}(p(1), u) - e^{(q,N)}(q(1), u) \right| > 3\varepsilon,$$

then such a dataset would contradict the symmetry of the Slutsky matrix.

3. Answer the following questions.

   (a) State the definition of a supermodular function.

(b) Suppose that $(X, \leq_X)$ and $(Y, \leq_Y)$ are lattices. Describe a partial order on $X \times Y$ that makes it into a lattice. Carefully define the lattice operations on the product lattice.

(c) Suppose that $(X, \leq_X)$ is an arbitrary lattice and $Y = R_+$ is a lattice of positive real numbers. Suppose that $f : X \to R$ is supermodular and increasing. The latter means that for any $x \leq_X x'$, we have $f(x) \leq f(x')$. Show that function $g : X \times Y \to R$ defined so that

$$g(x, y) = yf(x)$$

is supermodular on the product lattice. Take $y, y' \in Y$ and $x, x' \in X$. We need to show that

$$g(x \vee x', \max(y, y')) + g(x \vee x', \min(y, y')) - g(x, y) - g(x', y') \quad \geq \quad 0.$$

Suppose without loss of generality that $y < y'$. Then, the left-hand side is equal to

$$y' f(x \vee x') + yf(x \wedge x') - yf(x) - y' f(x') = y'(f(x \vee x') - f(x)) - y(f(x) - f(x \wedge x')).$$

Because f is supermodular,

$$f(x \vee x') - f(x) \geq f(x) - f(x \wedge x').$$

Together with $y' > y$, we obtain

$$y' f(x \vee x') + yf(x \wedge x') - yf(x) - y' f(x') \geq 0.$$

4. Consider a decision maker with Bernoulli utility function $u(.)$ and expected utility preferences over lotteries F.

   (a) Define Arrow Pratt risk aversion $r_u(x)$ and certainty equivalent $c(u, F)$ of lottery F.

(b) Show that for any two increasing utility functions u and v, if $r_u(x) \leq r_v(x)$ for each x, then, for each F

$$c(u, F) \geq c(v, F).$$

Carefully describe the proof of the above claim.

(c) Recall that a lottery F is a c.d.f. of a probability distribution over prizes z. For each lottery F and each $a \in R$, let $F + a$ be a lottery obtained from F by adding a to each prize. In other words, $(F + a)(z + a) = F(z)$ for each z. Show that if $r_u(x)$ is strictly decreasing in x, then for each $a > 0$,

$$c(u, F) + a \leq c(u, F + a).$$

We can define utility function $v_a(x) = u(x + a)$. Then, $r_{v_a}(x) = r_u(x + a)$, which implies that $r_{v_a}(x) \leq r_u(x + a)$. The above claim implies that for each lottery F

$$c(u, F) \leq c(v_a, F) = c(u, F + a) - a.$$

(d) Use the above observation to comment about the change in risk-attitudes for different levels of wealth when the decision maker has (i) Cobb-Douglas utility function $u(c) = c^{\alpha}$ for some $\alpha \in (0, 1)$, or (ii) the exponential utility $v(x) = -e^{-ax}$. The risk-aversion for Cobb Douglass $r_u(x) = \frac{1-\alpha}{c}$ is decreasing. The risk-aversion for the exponential utility $r_v(x) = a$ is constant. The risk attitudes do not change for the latter. The Cobb-Douglas guy becomes less risk averse when he is wealthier.