Microeconomic Theory I
Midterm

November 3, 2016

Name: ....

Student number: ....

Q1	Points	Q2	Points	Q3	Points	Q4	Points
1a		2a		3a		4a	
1b		2b		3b		4b	
1c		2c				4c	
		2d				4d	

Each question has the same value. You need to provide arguments for each answer. If you cannot solve one part of the problem, don't give up and try to solve the next one. If the question explicitly asks you to prove a result from the class, you must carefully describe the proof. Otherwise, you may use any result from the class given that you clearly state the assumptions, thesis and verify that the assumptions hold in your application. You have 110 minutes. Good luck!

1. Let $X = R^2$ and let $\preceq$ be a rational preference relation on X.

   (a) Explain what it means that preferences $\preceq$ are monotonic and/or continuous.

   Preference relation is monotone, if for each $x, y \in X$, if $x \ll y$, then $x \prec y$.

   Preference relation is *continuous* if it is preserved under limits: for all convergent sequences $x^n \to x$ and $y^n \to y$, if $x^n \preceq y^n$ for all n, then $x \preceq y$.

(b) Give an example of preferences $\preceq$ that are monotonic but not continuous. Justify all your claims.

Lexicographic preferences: $(x, y) \prec (x', y')$ iff $x < x'$ or $x = x'$ and $y < y'$ are monotonic (by definition), but not continuous (say $\left(\frac{1}{n}, 0\right) \succ (0, 1)$ but $(0, 0) \prec (0, 1)$).

(c) Give an example of preferences $\preceq$ that do not have a utility representation. Justify all your claims. (You can use the same example as in case (a).)

Lexicographic preferences do not have utility representation. See class notes for teh proof (or MWG).

2. Suppose that the space state $S = \{s, t\}$ and the space of prizes $Z = \{0, 1\}$ consist of two elements. We assume that $\preceq$ is a rational and preference relation over AA acts (i.e., the Axiom 1 of the expected utility theory holds).

   (a) Use a well-labeled diagram to graphically represent all AA acts.

(b) Suppose that $\preceq$ satisfy the Independence Axiom, and that the preferences are non-trivial (i.e., there are acts f and f' such that $f \prec f'$). Explain that the indifference curve are parallel lines.

Let $I(f) = \{g : g \sim f\}$ ne the indifference curve of act f. For any two acts $f \neq g$, let $L(f, g) = \{\alpha f + (1-\alpha) g : \alpha \in \mathbb{R}\} \cap X$ b ethe intersection of line that passes through f and g and space of AA acts

- *Convex*: The Independence Axiom implies that the indifference curves are convex: for any $f \sim g$, any $\alpha \in [0,1]$, $f \sim g \sim \alpha f + (1-\alpha) g$.
- *Contain whole lines*: Take any $f \sim g$, $\alpha \notin [0,1]$, and suppose that $h = \alpha f + (1-\alpha) g$ is an act (i.e., for each state s and prize z, $(\alpha f + (1-\alpha) g)(z, s) > 0$). W.l.o.g. assume that $\alpha > 1$. Then, if $h \succ g$, the Independence axiom implies that $f = \frac{1}{\alpha} h + \left(1 - \frac{1}{\alpha}\right) g \succ g$, which contradicts $f \sim g$. Similarly, it cannot be that $g \succ h$. Hence, it must be that $h \sim f \sim g$. Thus, $L(f, g) \subseteq I(f)$.
- *Indifference lines are not thick*: For any$f \sim g$, $f \neq g$ take any $h \in \text{int}X$, but $h \notin L(f, g)$. We show that $h \notin I(f)$. Indeed, if $h \in I(f)$, then $I(f)$ contains a non-empty interior (draw a picture to convince yourself!). In particular, there is act $i \in \text{int}\Delta(f, g, h) \subseteq I(f)$.

  Take any act $j \in X$. Because i belongs to the interior of $I(f)$, there exists $j' \in I(f) \cap L(i, j)$ and $j' \neq i$. Notice that $j \in L(i, j')$. Because $i \sim j'$, the above argument implies that $j \sim i \sim j' \sim f$. Because the act j is arbitrary, it means that $I(f) = X$. But this contradicts the assumption that preferences are not trivial.
- *Parallel*: The fact that preferences are parallel is a consequence of the Independence and a bit of geometry.

Another application of the Independence Axiom shows that they must be parallel.

(c) Give an example of preferences that have the State-Dependent Expected Utility (SDEU) representation, but not the State-Independent Expected Utility (SIEU) representations. Illustrate your answer on a diagram.

Suppose that $u_s(0) < u_s(1)$ and $u_t(0) > u_t(1)$.

(d) For $x_1, x_2 \in \Delta Z$, let (x_s, x_t) be an act that gives prizes x_i in state i. Suppose that $\preceq$ satisfy the Continuity and the Independence axiom and that $(1,1) \prec (0,1) \prec (0,0)$. Show that $\preceq$ have the S**I**EU representation.

If the preferences satisfy the Continuity and Independence, they have SDEU representation u_i. By choosing appropriate intercept of the affine transformation, we can always normalize $u_s(0) = u_t(0) = 0$. Because of the assumption $(1,1) \prec (0,1) \prec (0,0)$, it must be that

$$u_s(1) + u_t(1) < 0 + u_t(1) < 0 + 0.$$

It follows that $u_s(1), u_t(1) < 0$.

Let $u(0) = 0$ and $u(1) = u_s(1)$ and take $\pi_s = \frac{1}{1+u_t(1)/u_s(1)}$ and $\pi_t = \frac{u_t(1)/u_s(1)}{1+u_t(1)/u_s(1)}$. Observe that the two representations are equivalent.

3. Consider an investor with the utility over final consumption given by function $u : \mathbb{R} \to \mathbb{R}$. We assume that u is twice continuously differentiable, increasing and concave and that

$$u'(x) + xu''(x) > 0 \text{ for each } x.$$

The investor chooses how much $a \in [0, w]$ of his wealth w to invest into the risky asset. The risky asset brings an uncertain return z. The safe asset yields a constant return r where $r > 0$.

(a) Let

$$v(a, z) = u(a(1+z) + (w-a)(1+r))$$

be the utility of the investor who invests a into the risky asset and the rate of return on the risky asset is z. Show that function v has increasing differences.

Observe that v is twice continuously differentiable. Let

$$x = a(1+z) + (w-a)(1+r) = a(z-r) + w(1+r).$$

Then,

$$v_{az} = (z-r)\,au''(x) + u'(x) = xu'' + u' - w(1+r)\,u'' > 0$$

because of the assumption and the fact that $u'' < 0$.

(b) Let $F_\gamma(z) = 1 - e^{-\gamma z}$ be a cumulative distribution function. Let a_γ be the optimal investment level if the uncertain return is distributed according to F_γ and the investor maximizes the expected utility. Show that a_γ is increasing in γ. Carefully state (and verify all the assumptions of) any result that you need to justify your claim.

There is a mistake in the question: a_γ is decreasing in γ

Notice that F_γ is ordered by the first-order stochastic dominance (the wrong way). Because of (a), the result in class implies that a_γ is decreasing in γ.

4. Answer the following questions:

(a) State the Laws of (Uncompensated) Demand.

Law of Demand:

$$\frac{\partial x_l\left(p,w\right)}{\partial p_l} \leq 0.$$

(In the answer, you should explain what x is - Walrasian demand).

(b) Use the Slutsky equation to discuss conditions which ensure that the Law of Demand holds.

Law of Demand:

$$\frac{\partial x_l\left(p,w\right)}{\partial p_l} \leq 0.$$

Slutsky equation (explain the terms!):

$$\frac{\partial x_l\left(p,w\right)}{\partial p_l} = \frac{\partial h_l\left(p,v\left(p,w\right)\right)}{\partial p_l} - \frac{\partial x_l\left(p,w\right)}{\partial w}\frac{\partial x\left(p,w\right)}{\partial p_l}$$

Observe that

$$\frac{\partial h_l\left(p,u\right)}{\partial p_l} \leq 0$$

because the Law of Compensated Demand always hold. If the wealth effects are always positive, then the Law of Demand holds as well..

(c) Explain that homothetic preferences satisfy the Law of Demand.

Homothetic preferences can be represented by utility function such that for each $\alpha > 0$, $u(\alpha x) = \alpha u(x)$. We will show that the wealth effect is always positive. Notice that $x(p, \alpha w) = \alpha x(p, w) = \alpha x\left(\frac{1}{\alpha}p, w\right)$. The second equality comes from homogeneity of degree 0 of Walrasian demand and the first equality. We will show the first. Indeed, suppose that $x \in x(p, w)$. Then,

$$\begin{aligned}
\arg\max_{x \in B(p,w)} u(x) &= \arg\max_{x \in B(p,w)} \alpha u(x) \\
&= \arg\max_{\alpha x \in \alpha B(p,w)} u(\alpha x) \\
&= \frac{1}{\alpha} \arg\max_{y \in \alpha B(p,w)} u(y) \\
&= \frac{1}{\alpha} \arg\max_{y \in B(p,\alpha w)} u(y)
\end{aligned}$$

which implies that

$$x(p, w) = \frac{1}{\alpha} x(p, \alpha w).$$

Denote $x^*(p) = x(p, 1)$. Then,

$$x(p, w) = w x^*(p)$$

Hence,

$$\frac{\partial x_l(p, w)}{\partial w} = x_l^*(p) \geq 0.$$

(d) Explain that quasi-linear preferences satisfy the Law of Demand.

Suppose that the preferences of the consumer are represented by quasi-linear utility

$$u(x) = \psi(x_1, ..., x_{L-1}) + x_L = \psi(\overline{x}) + x_L,$$

where we write $\overline{x} = (x_1, ..., x_{L-1})$. We assume that the consumption space is $X = R_+^{L-1} \times R$ (in particular, the consumption of the *numeraire* can be negative). Function ψ is continuous, strictly increasing, and strictly quasi-concave. Finally, we normalize the price of numeraire at $p_L = 1$.

We will show that the wealth effect is always zero.

Due to Walras Law, $x_L = w - \overline{p} \cdot \overline{x}$ and the consumer problem is equivalent to unconstrained problem :

$$\begin{aligned}
v(\overline{p}, w) &= \max_{\overline{x}, x_L : \overline{p} \cdot \overline{x} + x_L \leq w} \psi(\overline{x}) + x_L \\
&=^{\text{Walras Law}} \max_{\overline{x}, x_L : \overline{p} \cdot \overline{x} + x_L = w} \psi(\overline{x}) + x_L \\
&= \max_{\overline{x}} \psi(\overline{x}) + w - \overline{p} \cdot \overline{x} \\
&= w + \max_{\overline{x}} \psi(\overline{x}) - \overline{p} \cdot \overline{x}.
\end{aligned}$$

Let $\overline{x}(p)$ denote the solution to the above problem. Notice that the solution does not depend on p.

Due to No Excess Utility, the expenditure minimization can also be rewritten as an unconstrained problem:

$$\begin{aligned}
e(p, u) &= \min_{\overline{x}, x_L : \psi(\overline{x}) + x_L \geq u} \overline{p} \cdot \overline{x} + x_L \\
&= \min_{\overline{x}, x_L : \psi(\overline{x}) + x_L = u} \overline{p} \cdot \overline{x} + x_L \\
&= \min_{\overline{x}} \overline{p} \cdot \overline{x} + u - \psi(\overline{x}) \\
&= u + \min_{\overline{x}} \overline{p} \cdot \overline{x} - \psi(\overline{x}) \\
&= u - \max_{\overline{x}} \psi(\overline{x}) - \overline{p} \cdot \overline{x}.
\end{aligned}$$

Notice that $\overline{x}$ solves the above problem if and only if $x \in \overline{x}(\overline{p})$. In particular, the Walrasian and the Hicksian demends for non-numeraire goods are identical and there are no wealth effects.

(e)