Microeconomic Theory I
Midterm October 2017

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Each question has the same value. You need to provide arguments for each answer. If you cannot solve one part of the problem, don't give up and try to solve the next one. If the question explicitly asks you to prove a result from the class, you must carefully describe the proof. Otherwise, you may use any result from the class given that you clearly state the assumptions, thesis and verify that the assumptions hold in your application. You have 110 minutes.

An advice: It is always a good idea to start each "proof" answer with writing precisely what is that you want to show. It will help you to make it precise when thinking about solution. It will also allow me to give you a tiny bit of partial credit if the rest of the answer turns out to be wrong.

Good luck!

1. *Quasi-linear and homothetic preferences.* Consider a consumer with Walrasian demand $x(p,w) \in \mathbb{R}^L$. Let

$$D_p x = \begin{bmatrix} \frac{\partial x_1}{\partial p_1} & \frac{\partial x_1}{\partial p_2} & & \frac{\partial x_1}{\partial p_L} \\ \frac{\partial x_2}{\partial p_1} & \frac{\partial x_2}{\partial p_2} & & \frac{\partial x_2}{\partial p_L} \\ & & & \\ \frac{\partial x_L}{\partial p_1} & \frac{\partial x_L}{\partial p_2} & & \frac{\partial x_L}{\partial p_L} \end{bmatrix}$$

be the matrix of the derivatives of x.

(a) State the Slutsky equation and briefly (no more than 1 sentence) describe each term.

$$\frac{\partial h_l(p,u)}{\partial p_k} = \frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w} x_k(p,w),$$

or

$$\frac{\partial x_l(p,w)}{\partial p_k} = \frac{\partial h_l(p,u)}{\partial p_k} - \frac{\partial x_l(p,w)}{\partial w} x_k(p,w).$$

The Slutsky equation decomposes the Walrasian demand into the change into the Hicksian demand (compensated to attain the same utility level) and the wealth effect.

(b) Show that if the consumer's preferences are quasi-linear in good L, the restriction of the matrix $D_p x$ to the first $L-1$ terms is symmetric and positive semi-definite.

Suppose that the preferences of the consumer are represented by quasi-linear utility

$$u(x) = \psi(x_1, ..., x_{L-1}) + x_L = \psi(\overline{x}) + x_L,$$

where we write $\overline{x} = (x_1, ..., x_{L-1})$. We assume that the consumption space is $X = R_+^{L-1} \times R$ (in particular, the consumption of the *numeraire* can be negative). Function ψ is continuous, strictly increasing, and strictly quasi-concave. Finally, we normalize the price of numeraire at $p_L = 1$.

Due to Walras Law, $x_L = w - \overline{p} \cdot \overline{x}$ and the consumer problem is equivalent to unconstrained problem :

$$\begin{aligned} v(\overline{p}, w) &= \max_{\overline{x}, x_L : \overline{p} \cdot \overline{x} + x_L \leq w} \psi(\overline{x}) + x_L \\ &=^{\text{Walras Law}} \max_{\overline{x}, x_L : \overline{p} \cdot \overline{x} + x_L = w} \psi(\overline{x}) + x_L \\ &= \max_{\overline{x}} \psi(\overline{x}) + w - \overline{p} \cdot \overline{x} \\ &= w + \max_{\overline{x}} \psi(\overline{x}) - \overline{p} \cdot \overline{x}. \end{aligned}$$

Let $\overline{x}(p)$ denote the solution to the above problem. Notice that the solution does not depend on w.

It follows that the wealth effect term disappears and the $D_p \bar{x} = D_p h$. The latter matrix is symmetric as and negative semi-definite as the second derivative matrix of the convex expenditure function.

(c) Show that if the consumer's preferences are homothetic, the matrix $D_p x$ is symmetric and positive semi-definite.

Suppose that the consumer has homothetic preferences, ie. the preferences can be represented by utility function such that for each $\alpha > 0$, $u(\alpha x) = \alpha u(x)$. Then,

- $x(p, \alpha w) = \alpha x(p, w) = \alpha x\left(\frac{1}{\alpha}p, w\right)$. The second equality comes from homogeneity of degree 0 of Walrasian demand and the first equality. We will show the first. Indeed, suppose that $x \in x(p, w)$. Then,

$$\begin{aligned}
\arg\max_{x \in B(p,w)} u(x) &= \arg\max_{x \in B(p,w)} \alpha u(x) \\
&= \arg\max_{\alpha x \in \alpha B(p,w)} u(\alpha x) \\
&= \frac{1}{\alpha} \arg\max_{y \in \alpha B(p,w)} u(y) \\
&= \frac{1}{\alpha} \arg\max_{y \in B(p,\alpha w)} u(y)
\end{aligned}$$

  which implies that

$$x(p, w) = \frac{1}{\alpha} x(p, \alpha w).$$

- Denote $x^*(p) = x(p, 1)$. Then, $x(p, w) = w x^*(p)$.
- Slutsky equation:

$$\frac{\partial h_l(p, u)}{\partial p_k} = w \frac{\partial x_l^*(p)}{\partial p_k} + w x_l^*(p) x_k^*(p).$$

  In particular, matrix

$$D_p x = w D_p x^* = D_p h - w(x^*(p))(x^*(p))^T$$

  is symmetric and negative semi-definite.

2. *Non-linear budget sets.* In some countries, anti-poverty government programs provide for distribution of certain amounts $f_i \geq 0$ of basic goods (food, fuel, etc.) free to all individuals. If an individual wants to consumer more than the free amount, he or she can purchase the good on the market for price p_i. We can model the consumer's choice as a non-linear budget set:

$$B_i(p, w, f) = \left\{ x \in (\mathbb{R}_+)^L : \sum_i p_i \max(0, x_i - f_i) \leq w \right\}.$$

Here, L is the number of goods and $f = (f_1, ..., f_L) \in (\mathbb{R}_+)^L$ is the "free" bundle.

   (a) Suppose that $L = 2$, $f_1, f_2 > 0$, $p_1 = p_2$ and $w > 0$. Describe the budget set graphically on a diagram.

(b) From now on, assume that the consumer has a strictly monotone, strictly convex utility. Define the indirect utility in the standard way

$$v(p, w, f) = \max_{x \in B(p,w,f)} u(x).$$

Show that $v(., w, f)$ is quasi-convex.

Find $(p, w)$and (p', w') such that $v(p, w), v(p', w') \leq v^*$. Take any $\alpha \in [0, 1]$ .Let $x_\alpha \in x(\alpha(p, w) + (1 - \alpha)(p'w'))$. Then,

$$(\alpha p + (1 - \alpha) p') \cdot (x_\alpha - f)^+ \leq (\alpha w + (1 - \alpha) w').$$

This implies that either

$$p \cdot (x_\alpha - f)^+ \leq w,$$

or

$$p' \cdot (x_\alpha - f)^+ \leq w'$$

W.l.o.g. suppose the latter. In such a case, x_α is an affordable bundle at (p, w), and

$$v(\alpha(p, w) + (1 - \alpha)(p'w')) = u(x_\alpha) \leq u(x(p, w)) \leq v^*.$$

(c) Define the expenditure function as

$$e(p, u, f) := \min_x \sum_i p_i \max(0, x_i - f_i) \text{ st. } u(x) \geq u.$$

Show that $e(., u, f)$ is concave.

Take $\alpha \in [0, 1]$ and $h_\alpha \in h(ap + (1 - \alpha)p', u)$. Then, $u(h_\alpha) \geq u$ and h_α is one of the bundles that can be chosen in the dual problems (p, u) and $(p'u)$. In particular,

$$p \cdot (h_\alpha - f)^+ \geq e(p, u) \text{ and } p' \cdot (h_\alpha - f)^+ \geq e(p', u).$$

This implies that

$$e(\alpha p + (1 - \alpha)p', u) = (\alpha p + (1 - \alpha)p') \cdot (h_\alpha - f)^+ \geq \alpha e(p, u) + (1 - \alpha) e(p', u).$$

3. *Investment choice.* Consider a consumer who lives for two periods. The consumer has fixed wealth w, no other source of income, and decides how much of his wealth to invest. The consumer maximizes the lifetime utility

$$\max_s u_1(w-s) + u_2(\gamma s)$$

where s is the choice level of savings and $\gamma > 0$ is the return rate on the investment.

   (a) Suppose that the second period utility u_2 is twice continuously differentiable, increasing and concave, and the Arrow-Pratt *relative* measure of risk aversion is strictly smaller than 1.

$$-\frac{x\left(u_2''(x)\right)}{u_2'(x)} \leq 1.$$

   Does function

$$u_2(\gamma s)$$

   have increasing differences in γ and s.

   The cross-partial derivative is equal to

$$\frac{\partial^2}{\partial s \partial \gamma} u_2(\gamma s) = \frac{\partial}{\partial \gamma} \gamma u_2'(\gamma s) = u_2'(\gamma s) + \gamma s u_2''(\gamma s)$$

(b) Conclude that the optimal level of savings changes monotonically with the return rate in the strong set order sense. (For full credit, do not rely on the results from the class.)

By the previous answer, we know that

$$V(s,\gamma) = u_1(w-s) + u_2(\gamma s)$$

has increasing differences in γ and s. Suppose that $s \in s^*(\gamma)$ and $s' \in s^*(\gamma')$ for some $\gamma < \gamma'$. We want to show that $\max(s,s') \in s^*(\gamma')$ and vice versa. If $s \leq s'$, there is nothing to prove. Suppose that $s > s'$. Then,

$$0 \leq V(s,\gamma) - V(s',\gamma) \leq V(s,\gamma') - V(s',\gamma'),$$

where the first inequality comes from the fact that s is the optimal choice at γ and the second one comes from the increasing differences and the fact that $s > s'$and $\gamma < \gamma'$. Simiarly, we show that $\min(s,s') \in s^*(\gamma)$.

(c) What is the comparative statics of the optimal savings if the relative measure of risk aversion is strictly larger than 1?

Then, $u(\gamma(-\theta))$ has increasing differences in γ and θ. It will follow that the optimal level of savings is decreasing with the return rate.

4. *Prize State Independence.* Consider an agent with preferences $\preceq$ over Anscombe-Aumann acts $f : S \to \Delta Z$, where S is the state space, and Z is the space of prizes. For any act f, any state s, and any prize z, let

$$f_s z = (f_1, f_2, ..., f_{s-1}, z, f_{s+1}, ..., f_n)$$

be an act obtained from f by replacing the state s lottery by prize z. Say that $\preceq$ satisfy **Prize State Independence** (PSI) if for any act f, any two states s, s', and any prizes z, z', we have

$$f_s z \preceq f_s z' \iff f_{s'} z \preceq f_{s'} z'.$$

(a) Explain the difference between the PSI and the State Independence (SI) axiom from the class.

The SI is stated in the same way, but for all lotteries, instead of only prizes.

(b) Describe a concrete example of State-Dependent Expected Utility (SDEU) preferences that violate PSI.

S=$\{0,1\}$, $Z=\{a,b\}$. $u_0(a)=1=1-u_0(b)$, $u_1(b)=1=1-u_1(a)$.

(c) Show that if $|Z| = 2$ (i.e., there are only two prizes), then the any SDEU preferences that satisfy PSI have State-Independent Expected Utility (SIEU) representation.

Let $\{u_s\}$ be the SDEU representation that satisfies PSI. WE will show that there exists $\pi \in \Delta S$ and $u : Z \to \mathbb{R}$ such that $\preceq$ has SIEU representation (π, u).

Let $Z\{a, b\}$ be the space of prizes. Because of the PSI, it must be either that (a) $u_s(a) > u_s(b)$ for all states s, (b) $u_s(a) < u_s(b)$ for all states s, or (c) $u_s(a) = u_s(b)$ for all states s. IN the latter case, there argument is simple (take any prob. distribution $\pi \in \Delta S$ and any constant utility function u. Cases (a) and (b) are analoguous. Assume w.l.o.g. (a). Take

$$\pi_s = \frac{u_s(a) - u_s(b)}{\sum_{s'} u_{s'}(a) - u_{s'}(b)} = A\left(u_s(a) - u_s(b)\right),$$

where we take $A = \sum_{s'} u_{s'}(a) - u_{s'}(b)$. Also, let $B = \sum_s u(b)$. Then, $\pi_s > 0$ and $\sum \pi_s = 1$. Let $u(a) = 1$and $u(b) = 0$. Then, for any act f,

$$\begin{aligned}
\sum_s \sum_z f(s, z)\, u_s(z) &= \sum_s \left(u(b) + f(s, a)\left(u_s(a) - u_s(b)\right)\right) \\
&= B + A \sum_s \pi_s f(s, a) \\
&= B + A \sum_s \pi_s \sum_z f(s, z)\, u(z).
\end{aligned}$$

There is an alternative inderect proof that with two prizes, PSI is equivalent to SI. Then, the theroem from the class implies that Axioms 1-3 plus PSI implies SIEU.

(d) Does the above observation remain true if $|Z| > 2$? Prove it or disprove it using a counterexample.

No. Let $S = \{0, 1\}$ and $Z = \{a, b, c\}$ and take

$$u_0(a) = 0, u_0(b) = 1, u_0(c) = 2,$$
$$u_1(a) = 0, u_1(b) = 1, u_1(c) = 100.$$

Then, SI will be violated. In state 0, the agent prefers price b to lottery $a^{2/3}c^{1/3}$, and she has opposite preferences in state 1.