Here, we will show that continuous and monotone preferences have a a continuous utility representation.

Proposition 1. *Suppose that rational preference* $\preceq$ *relation on* $X = R_+^L$ *is continuous and monotone. Then, there exists a continuous function* $u : X \to R$ *that represents* $\preceq$.

Proof. **Step 1.** *We are going to show that for each bundle* $x = (x_1, ..., x_L)$*, there exists* $a_x \in R$ *such that*

$$x \sim (a_x, ..., a_x).$$

Fix bundle x. Define set

$$A(x) = \{a \geq 0 : x \preceq (a, ..., a)\}.$$

Set $A(x)$ is nonempty (because, due to monotonicity, it contains $\max_l x_l$). Let

$$a_x = \inf A(x)$$

be the infimum of elements of $A(x)$. That means that

- there exists a sequence $a^n \searrow a_x$ (converging to a_x from above) such that $a^n \in A(x)$ for each x, and
- that for each $a < a_x$, $a \notin A(x)$.

Because of the former property, we have $(a_n, ..., a_n) \succeq x$, and continuity implies that

$$(a_x, ..., a_x) = \lim_n (a^n, ..., a^n) \succeq x. \tag{0.1}$$

Because of the latter property, if $a_x > 0$, there exists a sequence $a^n \nearrow a_x$ (i.e., converging to a_x from below) such that $(a_n, ..., a_n) \preceq x$. Similarly as previously, continuity implies that

$$(a_x, ..., a_x) = \lim_n (a^n, ..., a^n) \preceq x. \tag{0.2}$$

If $a_x = 0$, then, due to monotonicity, for any $x \in X$,

$$(a_x, ..., a_x) = (0, ..., 0) \preceq x. \tag{0.3}$$

In any case, (0.1) and either (0.2) or (0.3) imply that

$$x \sim (a_x, ..., a_x).$$

Step 2. *Let $u(x) = a_x$. We are going to show that the utility function $u(.)$ that was constructed in step 1 represents our preferences.* Indeed, notice that for each $x, y \in X$,

$$x \preceq y$$

(by transitivity) if and only if

$$(a_x, ..., a_x) \preceq (a_y, ..., a_y)$$

(by monotonicity) if and only if

$$a_x \leq a_y$$

(by the construction of the utility function)

$$u(x) \leq u(y)$$

.

Step 3. *We are going to show that the utility function $u(.)$ is continuous, or, equivalently, that for each $x^n \to x$, $a_{x^n} \to a_x$.* On the contrary, suppose that there exists a sequence $x^n \to x$, such that not $a_{x^n} \to a_x$. That means that there exists $\varepsilon > 0$ such that inifnitely elements of the sequence a_{x^n} does not belong to the set $(a_x - \varepsilon, a_x + \varepsilon)$.

Either of the two claims holds (possibly both of them):

- there exists inifnitely many elements of the sequence a_{x^n} such that $a_{x^n} \leq a_x - \varepsilon$, or
- there exists inifnitely many elements of the sequence a_{x^n} such that $a_{x^n} \leq a_x - \varepsilon$.

Without loss of generality, assume that the second claim holds (the proof in the first case is analoguous). We can find a sebsequence $n_k \to \infty$ when $k \to \infty$ such that for each k, $a_{x^{n_k}} \leq a_x - \varepsilon$. Thus,

$$x_{n^k} \sim (a_{x^{n_k}}, ..., a_{x^{n_k}}) \preceq (a_x - \varepsilon, ..., a_x - \varepsilon).$$

By continuity of the preferences, because $x_{n^k} \to x$, we have

$$x \preceq (a_x - \varepsilon, ..., a_x - \varepsilon).$$

But, by monotonicity of preferences,

$$(a_x - \varepsilon, ..., a_x - \varepsilon) \prec (a_x, ..., a_x) \sim x.$$

This leads to a contradiction $x \prec x$. $\square$