SEPARATING HYPERPLANE THEOREM

The material in this notes can be partailly found in MWG Appendix M.G. The following two results are closely related.

Theorem 1. *(Separating Hyperplane Theorem, th. M.G.2. from MWG) Let* $B \subseteq R^N$ *be a convex set and* $x \notin B$*. Then, there exists a* $p \in R^N$ *such that* $p \neq 0$ *and* $p \cdot x > \sup_{y \in B} p \cdot y$*.*

Theorem 2. *(Supporting Hyperplane Theorem, th. M.G.3) Suppose that* $B \subseteq R^N$ *is convex, and that* $x \notin intB$*. Then, there is a* $p \in R^N$ *such that* $p \neq 0$ *and* $p \cdot x \geq \sup_{y \in B} p \cdot y$*.*
If x *is an extreme point of* B *(i.e.,* x *cannot be represented as a convex combination of elements of* B*), then we can find* $p \neq 0$ *such that* $p \cdot x > p \cdot y$ *for each* $y \in B, y \neq x$

We are going to use the following application to the consumer demand theory:

Corollary 1. *Suppose that* $u(x)$ *is a continuous, strictly quasi-convcave, and monotonic (i.e., increasing) utility function. Then, for each* x*, there exists a price vector* $p \gg 0$ *and wealth level* $w = p \cdot x$ *such that* $\{x\} = x(p, w)$*.*

Proof. Fix x. Consider set $B = \{y : u(y) \geq u(x)\}$. Because the utility is strictly quasi-concave, (a) set B is convex, and (b) $x \in B$ is an extreme point of B (for the latter, notice that if $y \neq y'$ and $y, y' \in B$, then

$$u(\alpha y + (1-\alpha) y') > \alpha u(y) + (1-\alpha) u(y') \geq \alpha u(x) + (1-\alpha) u(x) = u(x),$$

so $x \neq \alpha y + (1-\alpha) y'$.) Becuase the utility is increasing, if $y \geq x$, then $y \in B$.

By the Supporting Hyperplane, there exists $q \neq 0$ such that $q \cdot x > q \cdot y$ for each $y \in B, y \neq x$. Let $p = -q$. Then, $p \cdot x < p \cdot y$ for each $y \in B, y \neq x$.

We are going to show that $p \gg 0$. Suppose that $p_l \leq 0$ for some l. For each $\varepsilon > 0$, let

$$y_\varepsilon = (x_1, ..., x_{l-1}, x_l + \varepsilon, x_{l+1}, ..., x_L).$$

Then, $p \cdot y_\varepsilon = p \cdot x + \varepsilon p_l \leq p \cdot x$. However, $y_\varepsilon \geq x$ and $x \neq y_\varepsilon$, which implies that $y_\varepsilon \in B$. This leads to a contradiction with $p \cdot x < p \cdot y$ for each $y \in B, y \neq x$.

Let $w = p \cdot x$. Then, $x \in B(p, w)$. Moreover, for each $y \neq x$ st. $u(y) \geq u(x)$, we have $y \notin B(p, w)$. It follows that $x(p, w) = \{x\}$. $\square$