ECO2100: Marriage matching

Marcin Pęski

University of Toronto

1/6/2015
Introduction

- Matching models
  - a particular class of allocation problems,
  - many of them are purely social, i.e. they describe relations between groups of people,
  - also economic allocations, like workers to jobs, students to schools, tenants to dorm rooms, etc.)

- Network models:
  - another type of allocation problems.
Introduction

Goals

Goals of the lecture:

- matching models
  - classic matching theory
  - other (quirky) matching models,
  - matching in large population,
  - dynamic matching models

- network models.
  - network formation,
  - games on networks
Introduction to matching

Model

- Model of one-to-one matching with strict preferences (Gale and Shapley (1962); see Roth and Sotomayor (1992) for discussion of early results)
- Finite sets of men $M$ and women $W$
  - men - women,
  - workers - firms,
  - students - schools, etc.
- Each man $m$ has strict preferences over women and staying single, $<_m$.
- Each woman $w$ has strict preferences over men and staying single, $<_w$. 
Preferences and the entire problem is purely ordinal,
  typically, without loss of generality to assume that people have utility functions,
    as long as the problem is finite,
    if the problem is infinite, utility functions are more natural,

If preferences are not strict, typically, we can break the indifferences.
  everything that we say for the basic model applies also to the model in which preferences are broken
Introduction to matching

Model: matching

- Matching market: sets of men, women, and their preferences.
- Matching is one-to-one correspondence $\mu : M \cup W \rightarrow M \cup W$ that has order 2 (i.e., $\mu(\mu(x)) = x$) and such that if $\mu(m) \neq m$, then $\mu(m) \in W$, and similarly for $w$.
  - some man and women can remain unmatched,
  - otherwise, each man can marry only one woman, and vice versa
  - each worker can have at most one job, and each job can have at most one worker filling it,
  - each student can take only one spot in school, and each spot can be occupied only with one student
**Introduction to matching**

**Model: stability**

**Definition**

Matching $\mu$ is *stable* if it is

- *individually rational*: for each ind. $x$, $x$ (weakly) prefers her match to being alone,

\[ x \preceq_x \mu(x), \]

- *pairwise stable*: no pair of men and women can block the matching: for each $m, w$,

\[ \text{either } w \preceq_m \mu(m) \text{ or } m \preceq_w \mu(w) \]

- *cooperative concept* (not blocked by single or two-person deviations)
Main Problems of Matching Theory
Model: Issues

1. Existence.
2. Efficiency.
5. Uniqueness.
Men-proposing algorithm (there is analogous women-proposing)

Each man has a list of of women from top to bottom.
Each woman has a notebook with all the offers she got.
The algorithm moves through rounds,
At the beginning of round 0
  each man start with the complete list of preferences,
  each woman starts with her own name in the notebook,
Gale-Shapley algorithm

- At each round,
  - each man approaches woman at the top of his list
    - or does not approach anybody, if staying alone is on top of his list,
  - each woman adds all her offers to the notebook and crosses out all the suitors except for the best offer,
    - possibly, the best offer is to stay alone,
  - each man that is rejected by his top woman crosses her out of her list,
Gale-Shapley algorithm

- At each round,
  - the men’s list grows shorter,
  - the women’s best current offer is getting better.
  - because men and women are finite, the algorithm must stop.

**Theorem**

*The matching obtained in the men-proposing deferred acceptance is stable.*
Gale-Shapley algorithm

Proof.

- Individual rationality is obvious (no man or no woman ever rejects herself if it is a best option left)
- No blocking pair $m$ and $w$.
  - suppose that man $m$ likes woman $w$ more than his match $\mu^*(m)$.
  - man $m$ must have approached woman $w$ during the process and clearly got rejected.
  - but it means that woman $w$ preferred her temporary match in that round.
  - but her current match is always (weakly) better than all previous, rejected ones.
  - hence, $w$ prefers her current match.
Main Problems of Matching Theory

Model: Issues

1. Existence,
2. Efficiency.
   2.1 men-proposing algorithm is the best stable matching for all men,
   2.2 it is also the worst for all women.
3. Structure of the set of stable matchings,
5. Uniqueness.
Gale-Shapley algorithm
Optimality

Definition
Stable matching $\mu$ is $M$-optimal, if for each man $m$, for each other stable matching $\mu'$, $\mu(m) \geq \mu'(m)$.

Theorem
The deferred acceptance matching $\mu_M$ is $M$-optimal.

- The alternative, women deferred acceptance matching is $W$-optimal.
Gale-Shapley algorithm
Optimality: Proof of Theorem 14

Say that woman \( w \) is achievable by \( m \) if there is a stable matching \( \mu \) such that \( \mu(m) = w \).

- we need to show that if \( \mu_M(m) \neq m \), then \( \mu_M(m) \) is the best achievable woman by \( m \).
- it is enough to show that during the deferred acceptance, man \( m \) is never rejected by an achievable woman.

By induction on the rounds of the deferred acceptance.

- suppose that no man was rejected by an achievable woman till round \( k \).
- suppose that \( m \) is rejected by \( w \). If she rejects him because she is better off single, then clearly \( w \) is unachievable for \( m \).
- otherwise, she rejects him in favor of \( m' \) who prefers \( w \) to all his other achievable women (notice that \( m' \) was so far rejected only by women that were unachievable for him).
- but then, in any stable match, if \( m \) is matched with \( w \), \( w \) prefers \( m' \) to \( m \) and \( m' \) prefers \( w \) to his match (which must be achievable by definition). But this leads to a blocking pair.
Main Problems of Matching Theory

Model: Issues

1. Existence.
2. Efficiency.
3. **Structure of the set of stable matchings.**
   3.1 stable matchings = core
   3.2 the set of stable matchings forms a lattice
   3.3 rural hospital theorem: the set of unmatched people is the same across all matchings
5. Uniqueness.
Set of stable matchings
Core (i.e., coalitional, not just pairwise, stability)

- Matching is *blocked by coalition* $A \subseteq M \cup W$ if there exists (sub)matching $\mu' : A \rightarrow A$ such that $\mu'$ is stable on $A$ and everybody on $A$ prefers $\mu'$ to $\mu$.
- By def., matching is stable if and only if it is not blocked by any coalition of size 1 or 2.
- Matching $\mu$ is in the core if it is not blocked by any coalition.

**Theorem**
The core is equal to the set of all stable matchings.

- Proof: If there is a deviating coalition, then clearly, there exists a blocking pair. But this means that the original matching wasn’t stable.
Set of stable matchings

Lattice structure

For any two stable matchings $\mu$ and $\mu'$ define function $\lambda$:

$\lambda(m) = \mu(m)$ if $\mu(m) >_{m} \mu'(m)$ and $\lambda(m) = \mu'(m)$ otherwise (thus, men is matched to his more preferred partner)

$\lambda(w) = \mu(w)$ if $\mu(w) <_{w} \mu'(w)$ and $\lambda(w) = \mu'(w)$ otherwise (woman gets worse partner).

We also denote function $\lambda$ as $\mu \lor_{M} \mu'$.

Theorem

$\lambda = \mu \lor_{M} \mu'$ is a stable matching. All men prefer $\lambda$ to either $\mu$ or $\mu'$ and all women prefer $\mu$ to $\lambda$ (or, $\mu'$ to $\lambda$).
Set of stable matchings

Lattice structure

- First, we show that $\lambda$ is a good matching.
  - suppose that $w = \lambda(m) = \lambda(m')$ for $m \neq m'$.
  - both men $m$ and $m'$ prefer $w$ to their matches under both matchings $\mu$ and $\mu'$. Moreover each man is matched with $w$ under one of the matchings.
  - w.l.o.g. suppose that $w$ prefers man $m$ to man $m'$ and suppose that $w$ is matched with $m'$ under matching $\mu'$.
  - because man $m$ prefers $w$ to $\mu'(m)$ and $w$ prefers $m$ to $\mu'(w) = m'$, $(m, w)$ is a blocking pair and $\mu'$ is not stable. Contradiction.

- Also, suppose that $w = \lambda(m)$ and $m' = \lambda(w)$. If $m' \neq m$, then it must be that
  - woman $w$ prefers $m$ to $m'$ and she is married to $m'$ under one of the stable matching (say $\mu$)
  - man $m$ prefers $w$ to $\mu(m)$ (notice that $m$ cannot be married to $w$ under $\mu$, because $m'$ is).
  - but then $(m, w)$ is a blocking pair for matching $\mu$ - contradiction.
Next, we show that $\lambda$ is stable.

- suppose that $w$ prefers man $m$ to $\lambda(w)$. Then, $w$ prefers $m$ to both $\mu(w)$ and $\mu'(w)$.
- but this means that man $m$ prefers $\mu(m)$ and $\mu'(m)$ to $w$. In particular, $m$ prefers $\lambda(m)$ to $w$. 
Set of stable matchings

Lattice structure

Corollary

The set of matchings form a lattice with the men optimal at the top and the woman optimal at the bottom. The lattice is distributive: (check in BOOK)
Theorem

The set of people who stay single is the same in all stable matchings.

- Let $M^m(\mu)$ be the set of married men in matching $\mu$. Then, for each stable matching $\mu$

  $$M^m \text{ (men optimal)} \supseteq M^m(\mu) \supseteq M^m \text{ (women optimal)}.$$

- otherwise, if man $m$ would have higher utility in a non-optimal matching, which is impossible.

- A similar (reverse) observation holds for women.

- But for each match

  $$|M^m(\mu)| = |W^m(\mu)|!$$
Main Problems of Matching Theory

Model: Issues

1. Existence.
2. Efficiency.
5. Uniqueness.
Strategy proofness

Matching mechanisms

- A matching mechanism is mapping that assigns a matching market \( MM = (\mathcal{M}, (\leq_m)_{m \in \mathcal{M}}, \mathcal{W}, (\leq_w)_{w \in \mathcal{W}}) \) with a matching in this markets.

- For example, deferred acceptance is one of possible matching mechanisms.

Definition
Mechanism is \textit{strategy-proof} if each matching market \( MM \) for each \( x \), each preference relation \( \leq' \) over \( \{x\} \cup Y \) (where \( Y = M \) is \( x \) is a woman, and vice versa)

\[
\mu^{MM'}(m) \leq_x \mu^{MM}(m)
\]

where \( MM' \) is a matching obtained from \( MM \) by replacing preferences \( \leq_x \) of \( x \) by \( \leq' \)

- another name is (ex post) incentive compatibility
Strategy proofness
No strategy-proof mechanism that always delivers stable matchings

Theorem
There exists no strategy-proof matching mechanism such that $\mu^{MM}$ is stable matching for each market $MM$.

Proof: It is enough to find a market such that for each stable matching, at least one man or woman would like to restate his/her preferences.
Strategy proofness
No strategy-proof mechanism that always delivers stable matchings

- Proof: Example:

\[ w_1 > m_1, w_2, w_2 > m_2, w_1, \]
\[ m_2 > w_1, m_1, m_1 > w_1, m_2, \]

- Two stable matchings (a) \((m_1, w_1)\) and \((m_2, w_2)\) and (b) \((m_1, w_2)\) and \((m_2, w_1)\)

- Suppose that the matching mechanism chooses (a).

- If woman \(w_2\) misreports preferences \(m_1\) (i.e., she prefers to stay single rather than marry \(m_w\)), then (b) is the only stable matching.

- But \(w_2\) prefers \(m_1\) to \(m_2\), so she is better off misstating her preferences.
Lemma

In men-proposing algorithm,

- truth-telling is a dominant strategy for men,
- if woman w misstates her preferences, and gains (improves her match under the men-proposing algorithm), then the outcome of the match is stable under original preferences.
Proof of the first part:

- the only way that man $m$ can affect the outcome is by not approaching woman $\mu_M(m)$ (i.e., woman to whom he is married under the men-proposing algorithm)
- and possibly, moving to some women below her.
- in this way, he may initiate a chain, in which woman $\mu_M(m)$ chooses next in line suitor.
- But in this case, all women either stay the same or marry up relative to matching $\mu_M$—so no better woman available to $m$ and $m$ cannot gain!
Strategy proofness
Truth telling in men-proposing algorithm

- For the second part, suppose that all men and all women but \( w \) state their preferences truthfully, and \( w \) does not.
- Let \( m = \mu_M(m) \) be her match under the men-proposing algorithm.
- The only way that she can affect the outcome of the algorithm is if she rejects \( m \) (either by stating that \( m \) is worse than her most recent engaged men or that \( m \) is not acceptable.) (She is not approached by anybody better than \( m \)
Strategy proofness
Truth telling in men-proposing algorithm

- Rejecting $m$ initiates a rejection chain - $m$ goes to his next favorite woman, he is either accepted or rejected, if the former $mm$ is replaced by rejected $m_1$, etc. The rejection chain ends with one of the following events:
  - man approaches single woman,
  - man decides that there are no more acceptable women,
  - man $m'$ approaches $w$. If $m'$ is worse than $m$, then $w$ must reject him to restart the chain (otherwise she ends up with an outcome worse than $m$).

- Suppose that the chain ends with woman $w$ getting a better man. Then, this man must prefer woman $w$ to any other available match (otherwise he would not approach her), and the resulting matching is stable (nobody else can improve their matches).
Corollary

Suppose that in some matching market, woman $w$ has at most one stable partner (there exists at most one man $m$ such that $w$ is married to $m$ in some stable matching). Then, $w$ does not have incentives to lie under men-proposing algorithm.

The next result shows that uniqueness is connected with incentives.
Corollary

*If there exists unique stable matching, truth-telling is a best response strategy in men-optimal or women-optimal algorithms.*

- truth-telling and uniqueness are strongly connected.
- Question: Is it possible to extend the above result to say that if all stable outcomes are close, then the incentives to lie are very limited?
Main Problems of Matching Theory
Model: Issues

1. Existence.
2. Efficiency.
5. Uniqueness.
Uniqueness
Common preferences (a.k.a. 1-dimensional case)

Theorem

Suppose that all men have the same preferences. Then, there is a unique stable matching.

- Because all men have the same preferences, we can arrange all women from the best to the worst.
- Consider matching, in which the best woman picks first, then the second woman picks avoiding the pick of the first, etc.
- It is easy to see that in the men-proposing or women-proposing algorithms select this matching.
Uniqueness
Special case (exactly 1-dimensional)

- Men and women are distributed on the interval \( m, w \in [0, 1] \).
- Utilities \( M(m, w) \) and \( W(m, w) \) are increasing: Joint marriage production function
  \[ f(m, w) = M(m, w) + W(m, w). \]
- Say that \( f \) is strictly supermodular, if for any \( m < m' \) and \( w < w' \)
  \[ f(m, w) + f(m', w') > f(m, w') + f(m', w). \]
Uniqueness

Top-match property

Definition

Matching market has *top-match property* if for each subset of men $M' \subseteq M$ and for each subset of women $W' \subseteq W$, there exists man $\hat{m} \in M'$ and woman $\hat{w} \in W'$ such that both are most favorite matches in the subsets (man $\hat{m}$ is the most favorite men of woman $\hat{w}$ among all men in set $M'$ and vice verse).

$$\hat{m} \in \arg \max_m$$

It is easy to see that top-match property (and strict preferences) implies uniqueness. Clark (2006) shows that this property is also necessary if we want to have unique matching in each subset.
URL http://www.bepress.com/bejte/contributions/vol6/iss1/art8
