UNIVERSITY OF TORONTO

Faculty of Arts and Science

December 2013 Final exam

Advanced Economic Theory, ECO326H1F

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Duration - 120 minutes

No Aids Allowed

There are three questions with total worth of 100 points. Read the questions carefully. You must give a supporting argument and an answer in words to get full credit. If you don't know the answer to any of the parts, try to solve the next one. You do not need to compute the exact values of algebraic formulas (for example, it is OK to say that $x = 2 * \frac{1}{5} * \sqrt{100}$ instead of $x = 4$.) You have 120 minutes.

Total pages (including the title page): 4

Total marks: 100

Good luck!

(1) In the Rubinstein bargaining model, there are two players who are trying to divide a pie of size 1. In each period $t \leq T$, one player offers a division of the pie $(x, 1-x)$, and the other player accepts (in which case the game ends with payoffs $\delta^t x$ for player 1 and $\delta^t (1-x)$ for player 2) or rejects, in which case either, if $t < T$, the game moves to the next period $t+1$, or, if $t = T$ the game ends with payoffs 0 for both players. In the model that you learned in the class, the players alternate between making offers. Here, we consider a modification of the Rubinstein bargaining model in which player 1 has an opportunity to make an offer twice as often as player 2.

(a) Find the subgame perfect equilibrium strategies and payoffs in the game with 1 period only ($T = 1$), in which player 1 makes the only offer.

Equilibrium strategies:

- player 1: offer $(1, 0)$,
- player 2: accept if $x \leq 1$, reject otherwise.

Payoffs: $(1, 0)$.

(b) Find the subgame perfect equilibrium payoffs in the game with $T = 2$ periods , in which player 1 makes the offer in the first and player 2 makes the offer in the second period.

Given the answer to the previous question, the SPE payoffs in case player 2 rejects player 1 offer in the first period are $(0, \delta)$ (remember about the discounting and the fact that we flip the roles of the two players). Thus, in the first period, player 2 is indifferent to accept or reject the offer $(1-\delta, \delta)$. Because player 1 wants to maximize his share of the pie, in the unique SPE, he offers $(1-\delta, \delta)$, and the offer is accepted.

(c) Find the subgame perfect equilibrium payoffs in the game with $T = 3$ periods , in which player 1 makes the offer in the first period and in the second period, and player 2 makes an offer in the last period.

Given the answer to the previous question, the SPE payoffs in case player 2 rejects player 1 offer in the first period are $(\delta(1-\delta), \delta^2)$ (remember about the discounting). Thus, in the first period, player 2 is indifferent to accept or reject the offer $(1-\delta^2, \delta^2)$. Because player 1 wants to maximize his share of the pie, in the unique SPE, he offers $(1-\delta^2, \delta^2)$, and the offer is accepted.

(d) Find the subgame perfect equilibrium payoffs in the game with $T = 4$ periods , in which player 2 makes the offer in periods $t = 1$ and $t = 4$ and player 1 makes an offer in periods $t = 2$ and $t = 3$.

Given the answer to the previous question, the SPE payoffs in case player 1 rejects player 2 offer in the first period are $\left(\delta\left(1-\delta^2\right), \delta^3\right)$ (remember about the discounting). Thus, in the first period, player 1 is indifferent to accept or reject the offer $(\delta-\delta^3, 1-\delta+\delta^3)$. Because player 2 wants to maximize his share of the pie, in the unique SPE, he offers $(\delta-\delta^3, 1-\delta+\delta^3)$, and the offer is accepted.

(e) (Extra credit) Find the equilibrium payoffs when $T = 3k$ and $k \to \infty$ with player 1 making offers in periods $t = 1, 2, 4, 5, 7, 8, ..., 3k-2, 3k-1$ and player 2 making offers in periods $t = 3, 6, 9, ..., 3k$.

(2) Two entrepreneurs simultaneously make investment decisions. Andy chooses the level of investment equal to $e_1 \geq 0$, $e_1 \in R$. His payoff is equal to

$$e_1 (1 + \theta e_2) - e_1^2.$$

Here, $\theta = 6$ is a demand parameter. Beth chooses between two investment levels $e_2 = 0$ or $e_2 = 1$. (In particular, it is not possible to choose for her any intermediate level of effort). Her payoffs are equal to

$$\begin{cases} e_1 - c, & \text{if } e_2 = 1, \\ 0, & \text{if } e_2 = 0. \end{cases}$$

Here $c = 3$ is a cost parameter.

(a) Show that there exists a (Nash) equilibrium in which Beth invests.

Beth invests in the equilibrium, $e_2^* = 1$, if $e_1 \geq c = 3$. On the other hand, if Beth invests, Andy's optimal effort is equal to

$$e_1^* = \frac{\theta e_2^* + 1}{2} = \frac{7}{2}.$$

(We can derive it through first -order conditions on the Andy's maximization problem.) Because $e_1^* > c$, Beth is best responding as well.

(b) Suppose that Beth has two possible cost types, $c_L = \frac{7}{3} < c_H = \frac{11}{3}$. Andy does not know the cost type of Beth and he assigns probability $p = \frac{1}{2}$ to the low cost type, so that $pc_L + (1 - p) c_H = c = 3$. Beth knows her cost type. Show that Beth does not invest at all in any Bayesian Nash equilibrium of the game with incomplete information.

First, notice that the Andy's expectation about Beth investment level cannot be higher than $e_2^{\max} = 1$, which implies that Andy's equilibrium effort cannot be higher than $e_1^{\max} = \frac{\theta e_2^{\max}+1}{2} = \frac{7}{2} < c_H$. But then, investing is not a best response for the high cost type of Beth.

Next, suppose that there is a Bayesian Nash equilibrium in which Beth high cost type does not invest. The Andy's expectation about Beth investment level cannot be higher than $e_2^{\max,2} = p$ (that is because the high cost type of Beth does not invest for sure), which implies that Andy's equilibrium effort cannot be higher than $e_1^{\max} = \frac{\theta e_2^{\max,2}+1}{2} = \frac{3+1}{2} = 2 < c_L$. Thus, investing is not a best response for the low type of Beth.

(c) You can interpret (a) as a game with incomplete information in which neither Beth nor Andy knows Beth's cost type and both of the players have the same beliefs p. Compare the answers to (a) and (b). Does providing Beth with more information about her cost type improves her equilibrium payoffs?

In part (b), Beth never invests and she always get a payoff of 0. In part (a), Beth gets (in expectation) a strictly positive payoff. Thus, she is better off in part (a) and worse off when provided with additional information.

(d) Finally, suppose that both Beth and Andy know Beth cost type. Show that there exists an equilibrium in which Beth makes some investment.

Using the same arguments as in (b), we show that Beth high cost type will never invest. On the other hand, if Andy knows the type of Beth, there is an equilibrium in which the low cost type of Beth invests and "low cost type" Andy chooses $e_1\,(\text{"low cost type"}) = \frac{\theta+1}{2}$ amount of effort.

(3) Consider a two-player two-period game in which in the first period, the players play the Prisoner's Dilemma

Pl. 1\Pl. 2	Cooperate	Defect
Cooperate	9,9	0,10
Defect	10,0	5,5

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and in the second period, the players play the following game

Pl. 1\Pl. 2	Normal	Crazy
Normal	0,0	-5,-10
Crazy	-10,-5	-10,-10

,

(a) Show that in any Subgame Perfect Equilibrium (SPE), no player Cooperates in the first period. Carefully explain the equilibrium strategies.

The second period game has a unique Nash equilibrium (Normal, Normal). In any SPE, the second period actions for both players must be Normal, after each history. Because the second period behavior does not depend on the history, the only SPE behavior in the first period must involve the play of the dominant action Defect.

(b) Explain that the two-period game has a Nash equilibrium (not necessarily SPE) in which the players Cooperate in the first period. Carefully explain the equilibrium strategies and show that they indeed form an equilibrium.

Consider the following strategy profile:

$$\sigma_i(\emptyset) = \text{Cooperate},$$

$$\sigma_i(h) = \begin{cases} \text{Normal}, & \text{if } h = CC, \\ \text{Crazy}, & \text{otherwise.} \end{cases}$$

Such a a profile leads to payoffs 9+0=9. Any deviation in the second period is either inconsequential (because it changes the behavior on the off-path history h') or decereases the payoff from 9 to 9-10 (because it changes the Normal behavior after history CC to the Crazy behavior). The maximum first-period gain from the first period deviation is equal to 10-9=1. In the same time, assuming that the opponent plays the equilibrium stratregies, such a deviation will lead the opponent to play Crazy in the second period, which reduces the second period payoffs by 5. The total effect of the deviation is not higher than 1-5=-4. Thus, such a deviation is not profitable.

(c) Suppose that instead of the game described above, the players play the Prisoner's Dilemma in the first period followed by the Battle of Sexes (with payoffs given in the table below)

Pl. 1\Pl. 2	Stadium	Opera
Stadium	5,3	0,0
Opera	0,0	3,5

in the second period. Show that there exists an SPE in which both players Cooperate in the first period. Carefully explain the equilibrium strategies and show that they indeed form an equilibrium.

Consider the following strategy profile:

$$\sigma_i(\emptyset) = \text{Cooperate},$$

$$\sigma_1(h) = \begin{cases} \text{Stadium}, & \text{if } h = CC, \\ S^{5/8}O^{3/8}, & \text{otherwise.} \end{cases}$$

$$\sigma_2(h) = \begin{cases} \text{Stadium}, & \text{if } h = CC, \\ S^{3/8}O^{5/8} & \text{otherwise.} \end{cases}$$

The behavior in each subgame forms a Nash equilibrium. The expected payoffs in the second periods are equal to

$$(5, 3)\text{ ,if } h = CC,$$
$$\left(\frac{15}{8}, \frac{15}{8}\right)\text{ ,otherwise.}$$

Because the gain from the first-period deviation, 1, is smaller than the resulting loss in the second period, $\geq 3 - \frac{15}{8} = \frac{9}{8} > 1$, the strategies form an equilibrium in the initial (grand) subgame. Thus, such a profile is a SPE.

Total pages: 4

Total marks: 100