UNIVERSITY OF TORONTO

Faculty of Arts and Science

December 2014 Final exam

Advanced Economic Theory, ECO326H1F

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Duration - 120 minutes

No Aids Allowed

There are three questions with total worth of 100 points. Read the questions carefully. You must give a supporting argument and an answer in words to get full credit. If you don't know the answer to any of the parts, try to solve the next one. You do not need to compute the exact values of algebraic formulas (for example, it is OK to say that $x = 2 * \frac{1}{5} * \sqrt{100}$ instead of $x = 4$.)

You have 120 minutes.

Good luck!

Total pages (including the title page): 5

Total marks: 100

(1) (35 points) In this question, we are going to consider a three-period version of the Stackelberg model. There are two firms. In the first period, firm 1 (The Leader) chooses its base production level $q_1^B \geq 0$. Second, firm 2 (the Follower) observes the level chosen by the Leader and chooses its own production capacity q_2. Finally, the Leader observes the choice made by the Follower and it can adjust its production level to $q_1 = q_1^B + q_1^A$, where q_1^A is the adjustment level chosen by the Leader. We assume that $q_1^A \geq -q_1^B$, so that $q_1 \geq 0$.

The Follower's payoff is equal to

$$q_2 \left(\alpha - c - q_1 - q_2\right).$$

The Leader's payoff is similar,

$$q_1 \left(\alpha - c - q_1 - q_2\right) - \left(q_1^A\right)^2 =$$
$$= \left(q_1^A + q_1^B\right)\left(\alpha - c - \left(q_1^A + q_1^B\right) - q_2\right) - \left(q_1^A\right)^2.$$

The only difference is that the Leader pays the adjustment costs.

(a) Describe all period 2 and period 3 histories (i.e., histories that players observe at the beginning, respectively, of periods 2 and periods 3). Which histories are subhistories, and which are terminal histories?

Period 2: q_1^B. All histories are subhistories.
Period 3: $\left(q_1^B, q_2\right)$. All histories are subhistories.

(b) In the next three questions, we are going to find the subgame perfect equilibrium behavior for both players. First, what is the optimal (ie., best response) behavior of the Leader in the last period, given the choices made by both players in the previous periods?

In the last periods, the Leader chooses q_1^A to maximize

$$\left(q_1^A + q_1^B\right)\left(\alpha - c - \left(q_1^A + q_1^B\right) - q_2\right) - \left(q_1^A\right)^2.$$

The first order conditions imply that

$$\alpha - c - 2\left(q_1^A + q_1^B\right) - q_2 - 2q_1^A = 0.$$

Thus,

$$q_1^A = \frac{1}{4}\left(\alpha - c - 2q_1^B - q_2\right) = \frac{1}{4}\left(\alpha - c - q_2\right) - \frac{1}{2}q_1^B.$$

In particular, the total production level of the Leader is

$$q_1 = \frac{1}{4}\left(\alpha - c - q_2\right) + \frac{1}{2}q_1^B.$$

(c) Given that the Follower anticipates the behavior of the Leader in the last period. what is the optimal behavior of the Follower?

The Follower payoff is equal to

$$\begin{aligned} & q_2\left(\alpha - c - \frac{1}{4}\left(\alpha - c - q_2\right) + \frac{1}{2}q_1^B - q_2\right) \\ =& q_2\left(\frac{3}{4}\left(\alpha - c\right) + \frac{1}{2}q_1^B - \frac{3}{4}q_2\right). \end{aligned}$$

The first order conditions imply that

$$\frac{3}{4}\left(\alpha - c\right) + \frac{1}{2}q_1^B - \frac{3}{4}q_2 - \frac{3}{4}q_2 = 0,$$

or

$$q_2 = \frac{1}{2}\frac{4}{3}\left(\frac{3}{4}\left(\alpha - c\right) + \frac{1}{2}q_1^B\right) = \frac{1}{2}\left(\alpha - c\right) + \frac{1}{3}q_1^B.$$

(d) What is going to be the base level of production chosen in the first period? (Hint: To give a complete answer, this question requires lots of calculations. For a partial credit, explain that the optimal decision of the Leader is a solution to a certain maximization problem and CAREFULLY describe that maximization problem.) The Leader's payoff is

equal to

$$\left(q_1^A + q_1^B\right)\left(\alpha - c - \left(q_1^A + q_1^B\right) - q_2\right) - \left(q_1^A\right)^2.$$

Substituting the optimal adjustment's decision leads to

$$\left(\frac{1}{4}(\alpha-c-q_2)-\frac{1}{2}q_1^B+q_1^B\right)\left(\alpha-c-\left(q_1^A+\frac{1}{4}(\alpha-c-q_2)-\frac{1}{2}q_1^B\right)-q_2\right)$$
$$-\left(\frac{1}{4}(\alpha-c-q_2)-\frac{1}{2}q_1^B\right)^2$$
$$=\left(\frac{1}{4}(\alpha-c-q_2)+\frac{1}{2}q_1^B\right)\left(\alpha-c-\frac{1}{4}(\alpha-c-q_2)-\frac{1}{2}q_1^B-q_2\right)-\left(\frac{1}{4}(\alpha-c-q_2)-\frac{1}{2}q_1^B\right)^2$$
$$=\left(\frac{1}{4}(\alpha-c-q_2)+\frac{1}{2}q_1^B\right)\left(\frac{3}{4}(\alpha-c-q_2)-\frac{1}{2}q_1^B\right)-\left(\frac{1}{4}(\alpha-c-q_2)-\frac{1}{2}q_1^B\right)^2.$$

Notice that the optimal behavior of the fFollower implies that

$$\alpha-c-q_2=\alpha-c-\frac{1}{2}(\alpha-c)-\frac{1}{3}q_1^B=\frac{1}{2}(\alpha-c)-\frac{1}{3}q_1^B.$$

Thus,

$$\left(\frac{1}{4}(\alpha-c-q_2)+\frac{1}{2}q_1^B\right)\left(\frac{3}{4}(\alpha-c-q_2)-\frac{1}{2}q_1^B\right)$$
$$-\left(\frac{1}{4}(\alpha-c-q_2)-\frac{1}{2}q_1^B\right)^2$$
$$=\left(\frac{1}{4}\left(\frac{1}{2}(\alpha-c)-\frac{1}{3}q_1^B\right)+\frac{1}{2}q_1^B\right)\left(\frac{3}{4}\left(\frac{1}{2}(\alpha-c)-\frac{1}{3}q_1^B\right)-\frac{1}{2}q_1^B\right)$$
$$-\left(\frac{1}{4}\left(\frac{1}{2}(\alpha-c)-\frac{1}{3}q_1^B\right)-\frac{1}{2}q_1^B\right)^2$$
$$=\left(\frac{1}{8}(\alpha-c)+\frac{5}{12}q_1^B\right)\left(\frac{3}{8}(\alpha-c)-\frac{3}{4}q_1^B\right)-\left(\frac{1}{8}(\alpha-c)-\frac{7}{12}q_1^B\right)^2.$$

We take the first order conditions in the above expression:

$$0=\frac{5}{12}\left(\frac{3}{8}(\alpha-c)-\frac{3}{4}q_1^B\right)-\frac{3}{4}\left(\frac{1}{8}(\alpha-c)+\frac{5}{12}q_1^B\right)-2\frac{7}{12}\left(\frac{7}{12}q_1^B-\frac{1}{8}(\alpha-c)\right)$$
$$=(\alpha-c)\left(\frac{5}{12}\frac{3}{8}-\frac{3}{4}\frac{1}{8}+2\frac{7}{12}\frac{1}{8}\right)-q_1^B\left(\frac{5}{12}\frac{3}{4}+\frac{3}{4}\frac{1}{8}+2\left(\frac{7}{12}\right)^2\right).$$

Thus, a 1-period equilibrium strategy of the Leader is

$$q_1^B=\frac{\frac{5}{12}\frac{3}{8}-\frac{3}{4}\frac{1}{8}+2\frac{7}{12}\frac{1}{8}}{\frac{5}{12}\frac{3}{4}+\frac{3}{4}\frac{1}{8}+2\left(\frac{7}{12}\right)^2}.$$

West=*Peace*	Develop	Abandon
Military option	-10	-2
Sanctions	-1	1
Trade	-2	3

,

West=*Regime Change*	Develop	Abandon
Military option	-1	2
Sanctions	-2	0
Trade	-2	0

,

TABLE 1. The payoffs of the Western governments.

(2) (35 points) As you write this exam, the US, the EU and Iran prepare for yet another round of nuclear talks in Oman. In this question, you will see a simple game with incomplete information that attempts to model some of the issues that are (might be) at the back of negotiators' minds.

We are going to assume that Iran must choose whether to abandon or continue developing its nuclear weapon program. The nuclear program is costly, but the government prefers to pay that cost in order to defend itself against possible military action. The West decides whether to lift or continue sanctions that currently cripple Iran's economy. Additionally, the West may choose to exercise military option.

Iran's government faces uncertainty about the goals of the Western governments. Specifically, they don't know whether the West wants *Peace* with Iran (with probability $p > 0$), or a *Regime Change* (with the remaining probability $1-p$). The West faces the uncertainty about how developed the Iran's nuclear program is, or, as we model it here, how costly would it be for Iran to build a nuclear weapon. The cost can be either high (c_{high}, with probability q), or low (c_{low}, with probability $1-q$).

The payoffs of the two different types of the Western governments are presented the tables 1. The payoffs of the Iran's government are given Table 2.

Below, we are going to find a Bayesian Nash equilibrium of the above game.

(a) Explain that the Peace type of the West does not have a dominant action. Does it have a dominated action?

No dominant actions, but Military Option is dominated.

Iran's payoffs	Develop	Abandon
Military option	$-2-c$	-10
Sanctions	$-1-c$	-1
Trade	$2-c$	2

,

TABLE 2. Iran's payoffs.

(b) What about the Regime Change type of the West? Does it have any dominant or dominated actions?6

Military Option is dominant, and the other actions are dominated.

(c) Use your answer to the previous questions to explain that in any equilibrium, the Western Regime Change type always chooses military option, and the Western Peace type chooses between Sanctions , Trade, or any mixed strategy that involves these two actions.

The answer follows from the fact that dominated actions cannot be played in equilibrium.

(d) Show that that for each cost type c, there exists a value $p^*(c)$ such that if

$$p > p^*(c),$$

then Iran chooses to Abandon nuclear program in equilibrium, and if

$$p < p^*(c),$$

then Iran chooses to Develop. Find the value of $p^*(c)$. (It is enough to give a formula, you don't need to compute it.) *Hint: Use your answer to question (c) to represent the strategy of the Western Peace type as $S^{\alpha}T^{1-\alpha}$. What is the Iran's expected payoff from Abandon? Develop?*

The payoff from Abandon is

$$p(\alpha(-1)+(1-\alpha)(2))+(1-p)(-10).$$

The payoff from Develop is

$$
\begin{aligned}
&p\left(\alpha\left(-1-c\right)+\left(1-\alpha\right)\left(2-c\right)\right)+\left(1-p\right)\left(-2-c\right).\\
=&p\left(\alpha\left(-1\right)+\left(1-\alpha\right)2\right)+\left(1-p\right)\left(-2\right)-c.
\end{aligned}
$$

Thus, Abandon is a best response if

$$
\begin{aligned}
&p\left(\alpha\left(-1\right)+\left(1-\alpha\right)\left(2\right)\right)+\left(1-p\right)\left(-10\right)\\
&-\left[p\left(\alpha\left(-1\right)+\left(1-\alpha\right)2\right)+\left(1-p\right)\left(-2\right)-c\right]\\
=&\left(1-p\right)\left(-8\right)+c\geq 0,
\end{aligned}
$$

or

$$p\geq p^{*}\left(c\right)=1-\frac{c}{8}.$$

(e) How does the threshold depend on the cost type? Explain.

The higher the cost, the lower is the threshold.
To explain it, notice that if Iran's cost is high, they are unlikely to develop the nuclear weapons, even if they believe that the West is likely to want the Regime Change and exercise military option.

(f) Assume that $p>p^{*}\left(c_{\text{low}}\right)$. Given the previous questions, what are the equilibrium actions of the Peace type of the West?

If $p>p^{*}\left(c_{\text{low}}\right)$, then both types of Iran will choose to Abandon the nuclear program. In such a case, the West Peace type's best response is to lift the sanctions and do Trade.

(3) (30 points) Two individuals $i = 1, 2$ engage in a favor-giving game. Each individual chooses effort level $e_i \geq 0$ that describes how nice he wants to be to his partner. The payoffs of individual i are equal to

$$e_{-i} - \frac{1}{2} e_i^2.$$

In particular, they are increasing in the effort of the partner and decreasing in one's own effort.

(a) Find a Nash equilibrium of the above game.

$e_i = 0$.

(b) Find action profile (e_1^*, e_2^*) that maximizes the sum of payoffs of the two players.

$e_i^* = 1$.

(c) Suppose that the favor-giving game is repeated twice, one after another. Describe all subgame perfect equilibria of the twice repeated game.

Both players choose $e_i = 0$ in each period. This follows from a result in the class.

(d) How will your answer change if the game is repeated 3 times?

The answer is the same.

(e) Suppose now that the individuals play infinitely repeated favor-giving game. The two individuals have a discount factor $\delta \in (0, 1)$. Consider the following version of a Grim Trigger strategy:

(i) each player starts with an action $e_i = \widehat{e} \geq 0$,

(ii) each player continues playing $e_i = \widehat{e}$ until at least one player chooses some other level of effort $e_i \neq \widehat{e}$ (i.e., until at least one player "deviates"),

(iii) after the deviation, each player chooses $e_i = 0$.

Show that for each $\delta > 0$, there exist $e_\delta > 0$ such that for each $\widehat{e} < e_\delta$, a profile of the Grim Trigger strategies is an equilibrium.

The Grim Trigger strategy profile is going to be Nash equilibrium, if

$$\frac{1}{1-\delta}\left(\widehat{e}-\frac{1}{2}\widehat{e}^2\right) \geq \widehat{e}.$$

(The left hand side is an equilibrium payoff, and the right hand side is the payoff from the most profitable deviation.)

The above condition is equivalent to

$$\widehat{e} \leq 2\delta =: e_\delta.$$

Total pages: 5

Total marks: 100