Midterm

Advanced Economic Theory, ECO326F1H

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October, 2015

There are three questions. Read questions carefully. You must give a supporting argument and an answer in words to get full credit. If you don't know the answer to any of the parts, try to solve the next one. You have 120 minutes.

(1) (25 points) Consider the following game:

	L	C	R
U	$3,5$	$4,3$	$5,1$
M	$3,1$	$3,5$	$3,2$
D	$4,2$	$2,3$	$2,4$

(a) Find all strictly dominated strategies. Action M is strictly dominated by $U^{1/2+\varepsilon}D^{1/2-\varepsilon}$ for some very small $\varepsilon > 0$. Nothing else is (Every other action is a best response against some pure belief).

(b) Find all actions that can be eliminated by iterated elimination of strictly dominated strategies. After the first round of elimination, we get

	L	C	R
U	$3,5$	$4,3$	$5,1$
D	$4,2$	$2,3$	$2,4$

In this game, none of the action is strictly dominated (action C is equivalent is not dominated by a mixed strategy $L^{\alpha}R^{1-\alpha}$, and all other actions are best responses against pure belief.)

(c) Find all pure and mixed strategy Nash equilibria. After the first round of elimination, we get

	L^{β}	C^{γ}	$R^{1-\beta-\gamma}$
U^{α}	$3,5$	$4,3$	$5,1$
$D^{1-\alpha}$	$4,2$	$2,3$	$2,4$

There are no pure strategy equilibria. For similar reasons, there are no equilibria, where player 1 is playing only U or only D, nor player 2 playing only L or only C, or only R. Consider a strategy $U^{\alpha}D^{1-\alpha}$. In order for player 2 to be indifferent between any pair of her actions, it must be that $\alpha = \frac{1}{3}$. In fact, if player 1 randomizes $U^{1/3}D^{2/3}$, then player 2 is indifferent among all actions.

In order for player 1 to be indifferent, it must be that

$$3\beta + 4\gamma + 5\left(1 - \beta - \gamma\right) = 4\beta + 2\gamma + 2\left(1 - \beta - \gamma\right).$$

(2) (40) Two firms cooperate on producing a final product (think a frankfurter and a bun). Each firm produces a part that is later combined with the other part and sold on the market. Each firm $i = 1, 2$, chooses individually (and separately from the other firm) the quality $q_i \geq 0$ of its production. The profits from final product depend on the qualities of both firms and they are equal to

$$\pi(q_1, q_2) = a(q_1 + q_2 - q_1 q_2),$$

where $a \in [0, 2]$. The two firms split the profits equally and each pays its own production cost. The final payoffs are equal to

$$\frac{1}{2}\pi(q_1, q_2) - c(q_i).$$

where $c(q)$ are the costs of producing with quality q.

(a) For questions (a)-(b), suppose that $c(q) = q^2$. Find the best responses and all Nash equilibria. The FOC:

$$\frac{1}{2}a(1 - q_{-i}) - 2q_i = 0.$$

Thus, the best response is

$$q_i^{BR}(q_{-i}) = \max\left(0, \frac{1}{4}a(1 - q_{-i})\right).$$

First, we show that there is no Nash equilibrium, in which a player chooses 0 quality. Indeed, if $q_{-i} = 0$, then $q_i^{BR}(0) = \frac{1}{4}a$, and $q_{-i}^{BR}\left(\frac{1}{4}a\right) = \frac{1}{4}a - \frac{1}{16}a^2 > 0$.

Thus, the Nash equilibrium must solve

$$q_i^* = \frac{1}{4}a\left(1 - q_{-i}^*\right) = \frac{1}{4}a\left(1 - \frac{1}{4}a(1 - q_i^*)\right).$$

There exists only one solution:

$$q_i^* = \frac{a}{a+4}.$$

(b) Do players have any (strictly) dominated strategies? (For partial credit, find actions that are never best responses to pure strategies.) Find all dominated strategies and show that they are dominated. Any action

that is larger than $q^* = \frac{1}{4}a$ is a never best response to pure strategies. We will show that $q > \frac{1}{4}a$ it is dominated by $\frac{1}{4}a$ for player 1. We compute the difference between the payoffs from both actions:

$$\begin{aligned}
&\frac{1}{2}\pi\left(q, q_2\right) - c\left(q\right) - \left(\frac{1}{2}\pi\left(q^*, q_2\right) - c\left(q^*\right)\right) \\
=&\frac{1}{2}\left(a\left(1 - q_2\right)\left(q - q^*\right)\right) - \left(q - q^*\right)\left(q + q^*\right) \\
=&\left(\frac{1}{2}a\left(1 - q_2\right) - \left(q + q^*\right)\right)\left(q - q^*\right).
\end{aligned}$$

The last expression is stictly negative due to the fact that because $q_2 \geq 0$, and $q > q^* = \frac{1}{4}a$, we have

$$\frac{1}{2}a\left(1 - q_2\right) - \left(q + q^*\right) < \frac{1}{2}a - \left(\frac{1}{4}a + \frac{1}{4}a\right).$$

This implies that

(c) From now on, suppose that $c\left(q\right) = \begin{cases} f + q^2, & \text{if } q > 0, \\ 0, & \text{if } q = 0, \end{cases}$ where $f > 0$ is a fixed cost of producing non-zero quality. Find conditions on parameters such that there is an Nash equilibrium in which both firms choose zero qualities. If both firms choose $q = 0$, their payoffs are equal to 0. If firm

1 switches to $q > 0$, its payoff is equal to

$$\frac{1}{2}aq_1 - q_1^2 - f.$$

The above is maximized for $q = \frac{1}{4}a$. So, if

$$\max_q \frac{1}{2}aq - q^2 - f = \frac{1}{2}a\left(\frac{1}{4}a\right) - \left(\frac{1}{4}a\right)^2 - f = \frac{1}{16}a^2 - f \leq 0,$$

the deviation is not a best response.

(d) Find condition on parameters, such that there exists an equilibrium with strictly positive qualities for both firms. If both firms choose positive quantities, the equilibrium qualities are equal to

$$q^* = \frac{a}{a+4}.$$

If the payoffs are positive,

$$\frac{1}{2}a\left(2q^* - q^{*2}\right) - q^{*2} - f \geq 0$$

then it is n equilibrium.

(e) Do there exist parameters for which there exists an equilibrium in which exactly one firm has a positive quality? There are no conditions of this form. Suppose that $q_1 = 0$ and $q_2 > 0$ is an equilibrium. Thus, the payoff of firm 2 is positive:

$$\frac{1}{2}aq_2 - q_2^2 - f \geq 0.$$

If firm 1 chooses instead the same quantity $q_1' = q_2$, its payoffs are going to be equal to

$$\frac{1}{2}a\left(q_2 + q_2 - q_2^2\right) - q_2^2 - f = \left[\frac{1}{2}aq_2 - q_2^2 - f\right] + \left[\frac{1}{2}a\left(q_2 - q_2^2\right)\right].$$

Notice that the first term is positive because it is equal to the current equilibrium payoffs of firm 2. For the second term, $\frac{1}{2}a \leq 1$ implies that

$$\frac{1}{2}a\left(q_2 - q_2^2\right) > \frac{1}{2}aq_2 - q_2^2 \geq f > 0.$$

(3) (35 points) A professor wrote two questions for the exam. There is a "boring" question and an "interesting" question. The professor is tempted to choose the "interesting" question but she is afraid that the question might be too difficult (or, it might be not, who knows?).

Instead of choosing herself, the professor decides to let each student choose individually. There are N students. Each student chooses between one of two questions (different students may make different choices). The choices are described in the table

	probability of correct answer	points for correct answer
"boring" question	$\frac{1}{2}$	$A + c\frac{n}{N}$,
"interesting" question	q	B,

Here, $q \in [0,1]$ is a probability with which a student answers the "interesting" question, B is the number of points awarded for the correct answer to the "interesting" question, $c > 0$ is a constant, and $A + c\frac{n}{N}$ is the number of points awarded for the correct solution to the "boring" question. The latter depends on (is increasing with)

$n =$ the number of students who choose the "interesting" question.

We assume that $A < 2B$. You will find the explanation for such a grading scheme below.

Assume that each student wants to maximize the expected number of points (i.e., (probability of correct answer) * (points for correct answer)).

(a) Find q_* such that a student finds the "boring" question as a strictly dominant action if and only if $q < q_*$ (or, in other words, if the "interesting" question is too difficult). Answering the boring question is strictly

dominant if it is a best response regardless of what other students do. Suppose that $n' \in \{0, ..., N-1\}$ other students answers the "interesting" question. Then, the boring question is strictly dominant if for each

$n' \in \{0, ..., N-1\}$,

$$\frac{1}{2}\left(A + c\frac{n'}{N}\right) \geq qB.$$

Let $q_* = \frac{1}{2}\frac{A}{B}$.

(b) Find q^* such that a student finds the "interesting" question as a strictly dominant action if and only if $q > q^*$ (or, in other words, if the "interesting" question is too easy). Answering the boring question is strictly

dominant if it is a best response regardless of what other students do. Suppose that $n' \in \{0, ..., N-1\}$ other students answers the "interesting" question. Then, the interesting question is strictly dominant if for each $n' \in \{0, ..., N-1\}$,

$$\frac{1}{2}\left(A + c\frac{n'}{N}\right) \leq qB.$$

Let $q^* = \frac{1}{2}\frac{A + c\frac{N-1}{N}}{B}$.

(c) Given that there are n' other students choosing the interesting question, find the best response function of a student with difficulty parameter q.

The interesting question is a best response if

$$\frac{1}{2}\left(A + c\frac{n'}{N}\right) \leq qB,$$

The boring question is a best response if

$$\frac{1}{2}\left(A + c\frac{n'}{N}\right) \geq qB.$$

(d) Assume that all N students have the same difficulty parameter $q \in [0, 1]$. Find the conditions that characterize the number of students n^* who choose the interesting question in equilibrium. If there are n^* students choosing the interesting question in equilibrium, then the interesting question is a best response if $n^* - 1$ other students choose it:

$$\frac{1}{2}\left(A + c\frac{n^* - 1}{N}\right) \leq qB,$$

and the boring question is a best response if n^* students choose the

$$\frac{1}{2}\left(A + c\frac{n^*}{N}\right) \geq qB.$$

So, it must be that

$$\frac{1}{2}\left(A + c\frac{n^* - 1}{N}\right) \leq qB \leq \frac{1}{2}\left(A + c\frac{n^*}{N}\right).$$

Any n^* that satisfies the above inequalities is good.

(e) Is the exam grading system fair to students? Use (d) to explain that if the number of students is large, than, in equilibrium, the expected number of points awarded to each student is (approximately) the same, regardless whether they choose the "boring" or "interesting" question.