# Midterm

Advanced Economic Theory, ECO326S1H

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There are three questions. Read questions carefully. You must give a supporting argument and an answer in words to get full credit. If you don't know the answer to any of the parts, try to solve the next one. You have 110 minutes.

(1) (30 points) Consider the following game:

| | $L$ | $C$ | $R$ |
|---|---|---|---|
| $U$ | $3,5$ | $3,1$ | $3,3$ |
| $M$ | $4,2$ | $2,4$ | $4,3$ |
| $D$ | $3,5$ | $5,1$ | $2,3$ |

(a) Find all never best responses. Be careful! *Solution:* Action $U$ is strictly dominated by $D^{1/2-\varepsilon}M^{1/2+\varepsilon}$ for some very small $\varepsilon > 0$. Nothing else is. or instance, (action $R$ is a best response against a belief $M^{2/3}D^{1/3}$).

(b) Find all actions that can be eliminated by iterated elimination of never best responses. *Solution:* After the first round of elimination, we get

| | $L$ | $C$ | $R$ |
|---|---|---|---|
| | | | |
| $M$ | $4,2$ | $2,4$ | $4,3$ |
| $D$ | $3,5$ | $5,1$ | $2,3$ |

In this game, none of the action is strictly dominated (action $R$ is a best response against a belief $U^{1/2}M^{1/2}$).

(c) Find all pure and mixed strategy Nash equilibria. *Solution:* After the first round of elimination, we get

| | $L^{\alpha}$ | $C^{\beta}$ | $R^{1-\beta-\gamma}$ |
|---|---|---|---|
| | | | |
| $M^{\alpha}$ | $4,2$ | $2,4$ | $4,3$ |
| $D^{1-\alpha}$ | $3,5$ | $5,1$ | $3,3$ |

There are no pure strategy equilibria. For similar reasons, there are no equilibria, where player 1 is playing only $D$ or only $M$, nor player 2 playing only $L$ or only $C$, or only $R$. Consider a strategy $M^{\alpha}D^{1-\alpha}$. In order for player 2 to be indifferent between any pair of her actions, it must be that

$$4\alpha + 3(1 - \alpha) = 2\alpha + 5(1 - \alpha),$$

or $\alpha = \frac{1}{2}$. In fact, if player 1 randomizes $M^{2/3}D^{1/3}$, then player 2 is indifferent among all actions.

In order for player 1 to be indifferent, it must be that

$$3\beta + 4\gamma + 5(1 - \beta - \gamma) = 4\beta + 2\gamma + 2(1 - \beta - \gamma).$$

(2) (30) Many courses at the University are offered in different versions, one of them is typically more advanced, or more accelerated than the other. Students have often a choice in which version to enroll. Here, we analyze this choice. There are $N \geq 50$ students. Each student has ability $g_i \in [0,1]$, and we assume that $g_1 < ... < g_N$. Each student chooses whether to take course version $a_i = 1$ or $a_i = 2$. (Here, $a_i$ denotes $i$'s action..) Student $i$'s payoff from class $k = 1, 2$ is equal to

$$u_i(k, a_{-i}) = x_k - \left(g_{\max}^k - g_i\right)$$

depends on how much she will learn $x_k$, and it decreases with the distance between her ability and the highest ability in the course, $g_{\max}^k - g_i$, where

$$g_{\max}^k = \max\{g_j : a_j = k\}.$$

is the highest ability among all students who enroll in course $k$. (This is because there is always a bit of possibility that the professor may curve the grade.) We assume that students learn more in course 2: $x_1 < x_2$.

(a) What is the payoff of he highest ability student from choosing course $k$? Explain that she or he has has a strictly dominant action. *Solution:*

The payoff for the highest ability student from choosing course $k$ is $x_k$. Hence, she will always go to the more advanced class.

(b) What is the payoff of he lowest ability student from choosing course $k$? From now on, assume that $x_2 - x_1 < g_N - g_1$. Show that the lowest ability student does not have a strictly dominant action. *Solution:* The payoff for the lowest ability student from choosing course $k$ is $x_k - \left(g_{\max}^k - g_i\right)$. If $x_2 - x_1 < g_N - g_1$, then course 2 is a best response if nobody else takes it, and course 1 is a best response if student $N$ takes course

(c) Show that there exists $1 \leq i^* < N$ such that all students $i \geq i^* + 1$ have a strictly dominant action, and all students $i \leq i^*$ do not. *Solution:* Let $i^*$ be the highest ability student such that

$$g_N - g_{i^*} \geq x_2 - x_1.$$

Such student is well-defined and $i \geq 1$ and $i < N$. Each student $i > i^*$prefers course 2 (even with the highest ability student) rather than course 1 (even if she is the smartest student in the class). This is because teh former is not higher than $x_2 - g_N + g_i$ and the latter is not lower than $x_1$.

For any other student, they prefer course 1 if they are the smartest kids in the class, and course 2 if nobody else takes it.

(d) Show that there is a unique equilibrium. Can the equilibrium be found be the iterated elimination of strictly dominated actions? How many stages of elimination are required? *Solution:* All students $i \geq i^* + 1$ have a strictly dominant action to take course 2. None of the students $i \leq i^*$ does.

OTOH, if all students $i \geq i^*+1$ take course 2, then all students $i \leq i^*$ have a strictly dominant action to take course 1. indeed, the payoff difference is at most

$$x_1 - (g_{i^*} - g_i) - [x_2 - (g_N - g_i)] = x_1 - x_2 - g_{i^*} + g_N > 0.$$

Hence, two stages of elimination.

(3) There are $N$ citizens in the country. Each of the citizens chooses the level of environmentally negative activity $a_i \geq 0$. The activity yields direct utility $ua_i - ca_i^2$, where $u > 0$ is the utility and $c > 0$ is the cost, but it also has a negative environmental effect on the entire country (pollution, climate change, etc.). The payoff of each citizen is equal to

$$ua_i - \frac{1}{2}ca_i^2 - \frac{1}{2}\left(a_1 + ... + a_N\right)^2.$$

(a) Find a best response function. *Solution:* FOC:

$$u - ca_i - (a_1 + ... + a_N) = 0.$$

Hence,

$$a_i = \max\left(0, \frac{u - \sum_{j \neq i} a_j}{c+1}\right).$$

(b) Find the Nash equilibrium. Compute the equilibrium utility of each citizen $u_N^{eq}$ as a function of the size of the population $N$. *Solution:* $a^{eq} = \frac{u}{N+c}$. Everybody's utility is equal to

$$\frac{u^2}{N+c} - \frac{1}{2}\left(c + N^2\right)\left(\frac{u}{N+c}\right)^2 = \frac{u^2}{N+c}\left[1 - \frac{1}{2}\frac{N^2+c}{N+c}\right].$$

(c) Find the levels of activity that maximize the total welfare. Compare with the Nash equilibrium: in equilibrium, is there too much or too little of the activity? Compute the utility of each citizen $u_N^W$. *Solution:* The total welfare is equal to

$$u\sum a_i - \frac{1}{2}c\sum a_i^2 - \frac{1}{2}N\left(\sum a_i\right)^2.$$

FOC wrt $a_i$ lead to

$$u - ca_i - N\sum a_i = 0.$$

Hence, the welfare maximizing solution is symmetric and

$$a^W = \frac{u}{N^2+c} < a^{eq}.$$

The welfare of each citizen under welfare maximizing action is

$$u_N^W = \frac{u^2}{N^2+c} - \frac{1}{2}\left(c+N^2\right)\left(\frac{u}{N^2+c}\right)^2 = \frac{1}{2}\frac{u^2}{N^2+c}.$$

(d) The government is considering a tax on the activity $ta_i$ paid by everybody (i.e., carbon tax). The proceeds from the tax are immediately returned to citizens, in equal amounts per citizen. Thus, the payoff function changes to :

$$ua_i - \frac{1}{2}ca_i^2 - \frac{1}{2}\left(a_1 + ... + a_N\right)^2 - ta_i + t\frac{1}{N}\left(a_1 + ... + a_N\right).$$

Find the new equilibrium and equilibrium utility $u_N^{eq}(t)$ as a function of $t$. *Solution:* FOC:

$$u - \frac{N-1}{N}t - ca_i - \left(a_1 + ... + a_N\right) = 0.$$

Hence, the best response is

$$a_i = \max\left(0, \frac{u - \frac{N-1}{N}t - \sum_{j\neq i} a_j}{c+1}\right).$$

The equilibrium activity

$$a^{eq}(t) = \frac{u - \frac{N-1}{N}t}{N+c}.$$

The equilibrium utility

$$\begin{aligned} u_N^{eq}(t) =& \frac{u - \frac{N-1}{N}t}{N+c}\left(u - \frac{1}{2}c\frac{u - \frac{N-1}{N}t}{N+c} - \frac{1}{2}N^2\frac{u - \frac{N-1}{N}t}{N+c}\right) \\ =& \frac{u - \frac{N-1}{N}t}{N+c}\left(u - \frac{1}{2}\left(u - \frac{N-1}{N}t\right)\frac{N^2+c}{N+c}\right) \end{aligned}$$

(e) Find the welfare-maximizing level of tax. (To simplify the calculations, you can assume that the population is very large, i.e., $N \to \infty$, in which case,

$$u_N^{eq}(t) \to -\frac{1}{2}\left(u-t\right)^2.)$$

Compare with the answer that you found in question (c).

(f) Would your answer to the previous question changed if the government did not return the carbon tax proceeds to the citizens but used it to finance other programs?