ECO421: Beliefs

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Plan

Introduction

- Last time, we learned how to represent knowledge and interactive knowledge.
- The knowledge spaces are not enough to do game theory.
- The reason is that we cannot use the knowledge space to compute expected payoffs.
- For this, we need beliefs.

Plan

Introduction

- Type space:
  - Ω - space of states of the world
  - $t_i \in T_i$ - types for each player i
  - beliefs $\mu_i(\omega, t_{-i}|t_i)$.

Type space

- Each player type has a probability distribution over states of the world and types of the other player(s).
- Belief function of player i, type t_i
  - $\mu_i(\omega, t_{-i}|t_i)$ - probability that player i type t_i assigns to the state of the world being ω and the opponents having type t_{-i},
  - $\mu_i(\omega, t_{-i}|t_i) \geq 0$
  - $\sum_{\omega, t_{-i}} \mu_i(\omega, t_{-i}|t_i) = 1$ for each t_i.

Type space

Example: Investigation

- Trump has two types $T_{\text{Trump}} = \{G, I\}$.
- Trump's beliefs
  - if Trump is innocent, he knows for sure that the state of the world is $\omega = i$.
  - otherwise, he is not sure whether it is *ge* or *gn*:

$t_T \backslash \omega$	$\omega = i, t_M = NE$	$\omega = gn, t_M = NE$	$\omega = ge, t_M = E$
G	0	$1 - p$	p
I	1	0	0

.

- For instance,

$$\mu(\omega = gn | G) = 1 - p.$$

Type space

Relation between type space and knowledge space

- Each type space is also a knowledge space
- Take $\Omega^* = \Omega \times T_1 \times ... \times T_n = \{(\omega, t_1, ..., t_n) : \omega \in \Omega, t_i \in T_i \text{for each } i\}$
- Types
  - $T_i(\omega, ..., t_n) = \{(\omega', t'_1, ..., t'_n) : t'_i = t_i\}$
  - set of all tuples with type t_i
  - partitions Ω^*.

Plan

Prior and interim beliefs

- The beliefs $\mu_i(.|t_i)$ in the type space are called *interim*.
  - they are formed after the player learns his type.
  - also called *posterior*, but I prefer to use the latter term for the beliefs after the players learns the types of all other players (sometimes it is useful).
- Where do beliefs come from?

Prior and interim beliefs

Origin of beliefs: Two approaches

- *interim*: players start the game with some beliefs
- *ex ante:* players start with a prior
  - prior is a belief before they even realized they will play the game (model of the world),
  - they receive new information (type),
  - update their beliefs through Bayes formula.

Prior and interim beliefs

- A and B are two events.
- $P(A,B)$ - prior probability of event A and B.
  $P(A,B^c)$ prior probability of event A and not B.
  $P(A)$ - prior probability of event A.

$$P(A) = P(A,B) + P(A,B^c).$$

- Conditional probability of A given that B is true

$$P(A|B) = \frac{P(A,B)}{P(B)} = \frac{P(A,B)}{P(A,B) + P(A^c,B)}.$$

Bayes formula.

Prior and interim beliefs

Bayes formula

- "Updating" is a part of decision theory that talks about how people use new information.
- Bayesian updating is intuitive,
- Also the optimal thing to do for the expected utility maximizer.
- However, people do not always use Bayes formula
  - behavioral economics teaches us that people are not expected utility maximizers.
- Anomalies, mistakes.

Prior and interim beliefs

Example: Mammograms

- 1% of women get breast cancer (and therefore 99% do not),
- 80% of mammograms detect breast cancer when it is there (and therefore 20% miss it)
- 10% of mammograms detect breast cancer when it's not there (and therefore 90.4% correctly return a negative result).
- So, how should one feel after positive outcome of a mammogram?

Prior and interim beliefs

Example: Mammograms

- $A = \text{Cancer}, B = \text{TestPositive}$,
- $P(A) = 0.01$ - probability of a breast cancer
- $P(B|A) = 0.8$ - probability that a mammogram detects breast cancer when it is there,
- $P(B|A^c) = 0.1$ - probability of a false positive,

Prior and interim beliefs

Example: Mammograms

- $A = \text{Cancer}, B = \text{TestPositive}$,
- $P(A) = 0.01$ - probability of a breast cancer
- $P(B|A) = 0.8$ - probability that a mammogram detects breast cancer when it is there,
- $P(B|A^c) = 0.1$ - probability of a false positive,
- Bayes formula

$$P(A, B) = P(A) * P(B|A) = 0.008,$$
$$P(A^c, B) = P(A^c) * P(B|A^c) = 0.099,$$

$$P(A|B) = \frac{P(A, B)}{P(A, B) + P(A^c, B)} = \frac{0.008}{0.008 + 0.099} \approx 8\%.$$

- Positive test increases the likelihood of cancer from 1% to 8%.
- Mammograms are not very effective screening tool.

Prior and interim beliefs

Example: Mammograms

- For a more striking example of use of Bayes formula, check out
  - Oj Simpson trial.

Prior and interim beliefs

- Ex ante approach to the origin of beliefs.
- prior belief μ
  - μ is a probability distribution over (ω, t_i, t_{-i})
  - in particular, it says "how likely is that I will learn that my type is t_i"

$$\mu(t_i) = \sum_{\omega, t_{-i}} \mu_i(\omega, t_{-i}, t_i),$$

Prior and interim beliefs

- Ex ante approach to the origin of beliefs.
- prior belief μ
  - μ is a probability distribution over (ω, t_i, t_{-i})
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$$\mu(t_i) = \sum_{\omega, t_{-i}} \mu_i(\omega, t_{-i}, t_i),$$

- players learn their type t_i
- and update their beliefs through Bayes formula

$$\mu_i(\omega, t_{-i}|) = \frac{\mu(\omega, t_i, t_{-i})}{\mu(t_i)}.$$

Prior and interim beliefs

Example: Investigation

- R. Mueller started the investigation with a "prior" belief
  - $\mu(G) = q$ - probability that Trump is guilty,
  - $\mu(I) = 1 - \mu(G) = 1 - q$ probability that Trump is innocent.
- Mueller anticipates that
  - if Trump is innocent, evidence will never be found,
  - if Trump is guilty, evidence will be found with probability p.

Prior and interim beliefs

Example: Investigation

- Formally,
  - $\mu(I) = 1 - \mu(G) = 1 - q$,
  - $\mu(NE|I) = 1 - \mu(E|I) = 1$.
  - $\mu(E|G) = 1 - \mu(NE|G) = p$.
- It follows that

$$\begin{aligned}
\mu(G, E) &= \mu(G)\,\mu(E|G) = qp, \\
\mu(G, NE) &= \mu(G)\,\mu(NE|G) = q(1-p), \\
\mu(I, E) &= \mu(I)\,\mu(E|I) = (1-q)\,0 = 0, \\
\mu(I, NE) &= \mu(I)\,\mu(NE|I) = (1-q)\,1 = 1-q,
\end{aligned}$$

Prior and interim beliefs

Example: Investigation

- As a result of his investigation, Mueller updates his belief.
- If he finds evidence, to

$$\mu(G|E) = \frac{\mu(G,E)}{\mu(G,E) + \mu(I,E)} = \frac{\mu(G)\,p}{\mu(G)\,p + 0} = 1,$$

$$\mu(I|E) = \frac{\mu(I,E)}{\mu(I,E) + \mu(G,E)} = 0.$$

If he does not find evidence to

$$\mu(G|NE) = \frac{\mu(G)(1-p)}{\mu(G)(1-p) + (1-\mu(G))\,1} = \frac{q-qp}{1-qp}.$$

- Mueller beliefs about Trump:

$t_{\text{Mueller}} \backslash t_{\text{Trump}}$	$t_{\text{Trump}} = I,$	$t_{\text{Trump}} = G,$
E	0	1
NE	$\frac{1-q}{1-qp}$	$\frac{q-qp}{1-qp}$

.

Plan

Bayesian games

Definition of the game

- Players $i = 1, .., N$.
- Actions $a_i \in A_i$
- Type space $(\Omega, (T_i), \mu_i)$
- Payoffs

$$u_i (a_i, a_{-i}, t_i, t_{-i}, \omega).$$

- (Pure) strategy $\sigma_i : T_i \to A_i$
  - $\sigma_i (t_i)$ - action player by player i type t_i,
  - mixed strategy $\sigma_i : T_i \to \Delta A_i$, where ΔA_i is the set of probability distribution over actions (i.e., mixed actions).

Bayesian games

Definition of the game

- Expected payoff of player i type t_i who chooses a_i give opponent's strategy σ_{-i}:

$$U_i\left(a_i,\sigma_{-i},t_i\right)=\sum_{t_{-i},\omega} u_i\left(a_i,\sigma_{-i}\left(t_{-i}\right),t_i,t_{-i},\omega\right)\mu_i\left(t_{-i},\omega|t_i\right).$$

  Expectation is computed with respect to the interim beliefs.

- Action a_i is a *best response* for type t_i against σ_{-i} if for any other action a_i',

$$U_i\left(a_i,\sigma_{-i};t_i\right)\geq U_i\left(a_i',\sigma_{-i};t_i\right),$$

- Action a_i is *(strictly) dominated* for type t_i if there exists an action a_i' such that for all strategies of the other players σ_{-i},

$$U_i\left(a_i,\sigma_{-i};t_i\right)< U_i\left(a_i',\sigma_{-i};t_i\right).$$

- A *Bayesian Nash equilibrium* is a profile of strategies $\sigma = (\sigma_1, ..., \sigma_I)$ such that for each player i type t_i, a (possibly, mixed) action $\sigma_i(t_i)$ is a best response for type t_i against σ_{-i}

Equilibrium

Example: Investigation

- Trump has two actions: Fire Mueller, Do nothing
- Mueller has two actions: Charge, Wait.
- Payoffs depend on whether Mueller has the evidence:

$\omega = GE$	C	W
F	-5,0	–5,0
DN	-20,1	0,-1

and

$\omega = I, GN$	C	W
F	-5,0	-5,0
DN	0,-1	0,1

Equilibrium

- Mueller strategy $m(.)$, tryump strategy $t(.)$.
- Mueller has a (weakly) dominant action to Charge if he has evidence and Wait if he does not,

$$m(E) = C, m(NE) = W$$

- If Trump is innocent, he also has a dominant action to Do Nothing.

$$t(I) = DN.$$

- What if he is guilty?

Equilibrium

- If guilty, Trump assigns probability

$$\mu(GE|G) = p$$

that Mueller has the evidence and that Mueller is going to Charge.

  - payoff from F:

$$-5,$$

  - payoff from DN:

$$p(-20) + 0(1-p).$$

  - If $p > \frac{1}{4}$, Trump is going to fire Mueller.

Equilibrium

- In our game, Trump is going to fire Mueller only if he is guilty.

Prior and interim beliefs

Example: Jury

- Defendant is either guilty or innocent $\omega \in \{G, I\}$, with prior probability of $\omega = G$ equal to π.
- Jurors are listening to the trial where evidence is presented.
- Each juror is independently interpreting the evidence heard during the trial.
  - If the defendant is guilty, a juror interprets the evidence as proof of guilt (g) with prob. p and with the remaining probability $1 - p$, as the proof of innocence.
  - If the defendant is innocent, the juror interprets the evidence as proof of innocence (i) with prob. q and with the remaining probability $1 - q$, as the proof of guilt.
- We assume that $p > 1 - q$.
  - the guilty signal is more likely if the defendant is guilty then when he is innocent.

Prior and interim beliefs

Example: Jury

- Formally,
  - $P(G) = 1 - P(I) = \pi$,
  - $P(g_i|G) = 1 - P(i_i|G) = p$,
  - $P(i_i|I) = 1 - P(g_i|I) = q$.
- It follows that

$$
\begin{aligned}
P(g_i, G) &= \pi p, \\
P(i_i, G) &= \pi(1-p), \\
P(g_i, I) &= (1-\pi)(1-q), \\
P(i_i, I) &= (1-\pi)q, \\
P(g_i, g_{-i}, G) &= \pi p^2, \\
P(g_i, i_{-i}, G) &= \pi p(1-p), \\
&\text{etc.}
\end{aligned}
$$

Prior and interim beliefs

Example: Jury

- Types $t_i \in \{g, i\}$.
- What is the conditional probability that the defendant is guilty?

$$P(G|g) = \frac{P(G,g)}{P(g)} = \frac{P(G,g)}{P(G,g) + P(I,g)}$$
$$= \frac{\pi p}{\pi p + (1-\pi)(1-q)} > \pi,$$

where the last inequality comes from

$$\pi p + (1-\pi)(1-q) < \pi p + (1-\pi) p = p.$$

- The inequality means that posterior belief after signal g that the defendant is guilty is higher than the prior belief. (Good.)

Prior and interim beliefs

Example: Jury

- Similarly,

$$P(G|i) = \frac{P(G,i)}{P(i)} = \frac{P(G,i)}{P(G,i) + P(I,i)}$$
$$= \frac{\pi(1-p)}{\pi(1-p) + (1-\pi)q} < \pi,$$

where the last inequality comes from

$$\pi(1-p) + (1-\pi)q > \pi(1-p) + (1-\pi)(1-p) = 1-p.$$

- Notice that

$$P(G|g) > \pi > P(G|i),$$

or the conditional probability that the defendant is guilty increases after guilty signal.

Equilibrium

Jury: single juror

- The juror chooses whether to acquit (A) or convict (C).
- Payoffs:
  - v is the cost of convincting an innocent person,
  - 1 is the cost of acquitting a guilty person.

Juror	$\omega = i$	$\omega = g$
A	0	-1
C	$-v$	0

Equilibrium

Jury: single juror

Juror	$\omega = i$	$\omega = g$
A	0	-1
C	$-v$	0

- Juror who believes that the defendant is guilty with prob. P chooses C if

$$-(1-P)\,v \geq -P, \text{ or}$$
$$v \leq \frac{P}{1-P}.$$

  Otherwise, A is the best response.
- That makes sense:
  - the higher the cost of wrong conviction, the less likely is the conviction.

Equilibrium

Jury: single juror

- Consider a juror who received a guilty signal g.
  Recall that

$$P(G|g) = \frac{\pi p}{\pi p + (1-\pi)(1-q)}.$$

- We check that the juror g chooses C if and only if

$$v \leq \frac{P(G|g)}{1 - P(G|g)} = \frac{\pi}{1-\pi}\frac{p}{1-q} = v_{\max}.$$

- That makes sense:
  - if the cost of if the prior belief that the defendant is innocent is very strong (π is small), the evidence is not going to convince the juror.

Equilibrium

Jury: single juror

- Similarly, if the juror receives signal i,

$$P(G|i) = \frac{\pi(1-p)}{\pi(1-p)+(1-\pi)q}.$$

Hence, the juror i chooses C if and only if

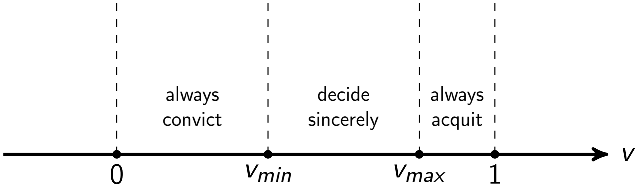
$$v \leq \frac{P(G|i)}{1-P(G|i)} = \frac{\pi}{1-\pi}\frac{1-p}{q} = v_{\text{min}}.$$

- Assumptions ($p > 1-q$) imply that $v_{min} < v_{\text{max}}$. Thus, if

$$v_{\text{min}} < v \leq v_{\text{max}}.$$

Equilibrium

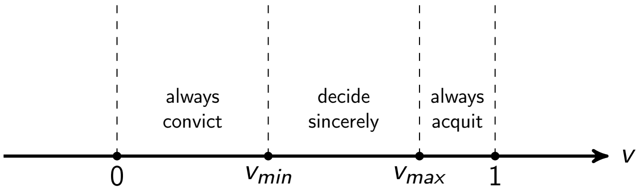
Jury: single juror



- The juror sentences sincerely only If the initial prior takes intermediate values $v \in (v_{min}, v_{max})$.
- Otherwise, the juror ignores the (imprecise) evidence.

Equilibrium

Jury: single juror



- Notice that values v_{min} and v_{max} depend on the prior belief.
- Convince yourself that if $\pi \to 1$, then $v_{min} \to \infty$ and $v_{max} \to \infty$.
  - What does it mean?

Equilibrium

Jury: two jurors

- Suppose that there are two jurors and both of them need to vote C in order to convict.
- Otherwise, if at least one says A, the defendant goes free.
- Payoffs are the same.
- Assume that juror $-i$ votes sincerely

$$s_{-i}(i) = A,$$
$$s_{-i}(g) = C.$$

- When is sincere voting a best response?

Equilibrium

Jury: two jurors

- Consider a juror with signal g.
- If the juror votes C, her payoff is

$$\begin{aligned}
&0P\left(\omega = G, t_{-i} = g | t_i = g\right) \\
&+(-v)\,P\left(\omega = I, t_{-i} = g | t_i = g\right) \\
&+(-1))P\left(\omega = G, t_{-i} = i | t_i = g\right) \\
&+0P\left(\omega = I, t_{-i} = i | t_i = g\right).
\end{aligned}$$

Equilibrium

Jury: two jurors

- Consider a juror with signal g.
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&+ 0P\left(\omega = I, t_{-i} = i | t_i = g\right).
\end{aligned}
$$

- If the juror votes A, her payoff is

$$
\begin{aligned}
&(-1)\,P\left(\omega = G, t_{-i} = g | t_i = g\right) \\
&+ 0P\left(\omega = I, t_{-i} = g | t_i = g\right) \\
&+ (-1))P\left(\omega = G, t_{-i} = i | t_i = g\right) \\
&+ 0P\left(\omega = I, t_{-i} = i | t_i = g\right).
\end{aligned}
$$

Equilibrium
Jury: two jurors

- Consider a juror with signal g.
- If the juror votes C, her payoff is

$$\begin{aligned}&\cancel{0P(\omega = G, t_{-i} = g | t_i = g)}\\&+(-v)P(\omega = I, t_{-i} = g | t_i = g)\\&\cancel{+(-1))P(\omega = G, t_{-i} = i | t_i = g)}\\&\cancel{+0P(\omega = I, t_{-i} = i | t_i = g)}.\end{aligned}$$

- If the juror votes A, her payoff is

$$\begin{aligned}&(-1)P(\omega = G, t_{-i} = g | t_i = g)\\&\cancel{+0P(\omega = I, t_{-i} = g | t_i = g)}\\&\cancel{+(-1))P(\omega = G, t_{-i} = i | t_i = g)}\\&\cancel{+0P(\omega = I, t_{-i} = i | t_i = g)}.\end{aligned}$$

Equilibrium

Jury: two jurors

- Thus, the juror will vote sincerely if

$$(-v)\, P\left(\omega = I, t_{-i} = g \middle| t_i = g\right) \geq (-1)\, P\left(\omega = G, t_{-i} = g \middle| t_i = g\right),$$

or

$$\begin{aligned} v &\leq \frac{P\left(\omega = G, t_{-i} = g \middle| t_i = g\right)}{P\left(\omega = I, t_{-i} = g \middle| t_i = g\right)} \\ &= \frac{P\left(\omega = G, t_{-i} = g, t_i = g\right) / P\left(t_i = g\right)}{P\left(\omega = I, t_{-i} = g, t_i = g\right) / P\left(t_i = g\right)} \\ &= \frac{P\left(\omega = G, t_{-i} = g, t_i = g\right)}{P\left(\omega = I, t_{-i} = g, t_i = g\right)}. \end{aligned}$$

Equilibrium

Jury: two jurors

- We compute

$$P\left(\omega = G, t_{-i} = g, t_i = g\right) = \pi p^2,$$
$$P\left(\omega = I, t_{-i} = g, t_i = g\right) = (1-\pi)\left(1-q\right)^2.$$

- Thus, the juror g will vote sincerely if

$$v \leq \frac{\pi}{1-\pi}\left(\frac{p}{1-q}\right)^2 = v_{max}^{(2)}.$$

- Notice that $p > 1 - q$ implies that $v_{max}^{(2)} > v_{max}$.

Equilibrium

Jury: two jurors

- What about voter i? If the juror votes C, her payoff is

$$
\begin{aligned}
&0P\left(\omega = G, t_{-i} = g | t_i = i\right) \\
&+ (-v)\, P\left(\omega = I, t_{-i} = g | t_i = i\right) \\
&+ (-1)\, P\left(\omega = G, t_{-i} = i | t_i = i\right) \\
&+ 0P\left(\omega = I, t_{-i} = i | t_i = i\right).
\end{aligned}
$$

- If the juror votes A, her payoff is

$$
\begin{aligned}
&(-1)\, P\left(\omega = G, t_{-i} = g | t_i = i\right) \\
&+ 0P\left(\omega = I, t_{-i} = g | t_i = i\right) \\
&+ (-1))P\left(\omega = G, t_{-i} = i | t_i = i\right) \\
&+ 0P\left(\omega = I, t_{-i} = i | t_i = i\right).
\end{aligned}
$$

Equilibrium

Jury: two jurors

- What about voter i? If the juror votes C, her payoff is

$$0P(\omega = G, t_{-i} = g | t_i = i)$$
$$+(-v)P(\omega = I, t_{-i} = g | t_i = i)$$
$$+(-1)P(\omega = G, t_{-i} = i | t_i = i)$$
$$+0P(\omega = I, t_{-i} = i | t_i = i).$$

- If the juror votes A, her payoff is

$$(-1)P(\omega = G, t_{-i} = g | t_i = i)$$
$$+0P(\omega = I, t_{-i} = g | t_i = i)$$
$$+(-1))P(\omega = G, t_{-i} = i | t_i = i)$$
$$+0P(\omega = I, t_{-i} = i | t_i = i).$$

Equilibrium

Jury: two jurors

- Thus, the juror i will vote sincerely if

$$v \geq \frac{P(\omega = G, t_{-i} = g, t_i = i)}{P(\omega = I, t_{-i} = g, t_i = i)}.$$

We compute

$$P(\omega = G, t_{-i} = g, t_i = i) = \pi p (1 - p),$$
$$P(\omega = I, t_{-i} = g, t_i = i) = (1 - \pi) q (1 - q).$$

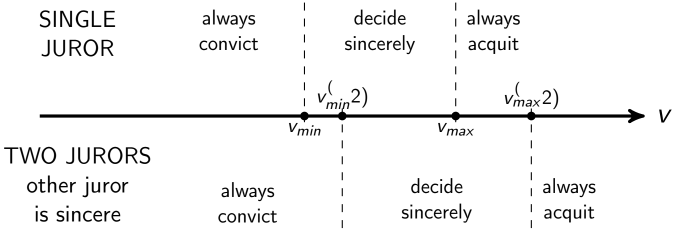
- Thus, the juror g will vote sincerely if

$$v \leq \frac{\pi}{1-\pi} \frac{p}{1-q} \frac{1-p}{q} = v_{min}^{(2)}.$$

- Again, $p > 1 - q$ implies that $v_{min}^{(2)} > v_{min}$.

Equilibrium

Jury: two jurors



- With two jurors, the thresholds are higher!
- Jurors feel happier to vote C! Why?

Equilibrium

Jury: two jurors

- With two jurors, the thresholds are higher!
- Jurors feel happier to vote C! Why?
- This is because the accused is sentenced only when the other juror $-i$ votes C.
- But if the other juror $-i$ is sincere, her voting C means that she got signal g.
- But that signal is informative. And juror i takes it into account when voting.

Plan

Conclusions

What did we learn - concepts

- Type space.
- Prior and interim beliefs. Bayes formula.
- Bayesian Nash equilibrium.

Conclusions

What did we learn - skills

- Represent a story with asymmetric information with a type space.
- Use the Bayes formula to find conditional probabilities.
- Derive interim beliefs from a prior.
- Find Bayesian Nash equilibria in games with incomplete information.