ECO421: Extensive-form games with incomplete information

Marcin Pęski

February 3, 2020

Plan

Introduction

Extensive-form games

Extensive form game with incomplete information

Perfect Bayesian equilibrium

Off-path beliefs

Conclusions

Introduction

- In a communication game from the last lecture, players choose their actions at different times:
  - first, the Sender chooses the message,
  - next, the Receiver chooses the action.
- Special case of extensive-form games, i.e., games that develop in time.
- Today, we remind the definition of the extensive form game, and
- formally introduce extensive-form games with incomplete information.

Plan

Introduction

Extensive-form games

Extensive form game with incomplete information

Perfect Bayesian equilibrium

Off-path beliefs

Conclusions

Extensive-form games

Histories

- History (at the beginning of period t) $h = (a_1, ..., a_{t-1})$,
  - $a_s = (a_{s,i}, a_{s,-i})$ is a profile of action in period $s < t$
  - histories describe what happened in the game.
- Histories consist of terminal histories and subhistories, $H = S \cup T$.

Extensive-form games

Types of histories

- Terminal histories: $h \in T$,
  - the game ends after such history,
  - payoffs $u_i(h)$.
- Subhistories: $h \in S$. Any proper subsequence $(a_1, ..., a_s)$ of a terminal history $(a_1, ..., a_t)$ where $s < t$ is called a *subhistory* (or, decision node).
  - $P(h)$ - set of players who moves after subhistory h,
  - $A_i(h)$ set of actions of player $i \in P(h)$,
  - Ex. initial history $h_0 \in S$.
  - each subhistory h induces a subgame h.

Extensive-form games

Definition

- Players $i = 1, .., I$
- Actions $A_i(h)$ for each $i \in P(h)$ and each subhistory h,
- Histories $H = T \cup S$
- Payoffs $u_i(z) \in R$ for each player i, for each terminal history $z \in T$,

Extensive-form games

Strategies

- A strategy of player i assigns action (or mixed action) to **each subhistory** at which the player makes a decision:

$$\sigma_i : \{h \text{ st. } i \in P(h)\} \to A_i.$$

  - a complete plan of action,
  - $\sigma_i(h)$ - an action of player i after history h st. $i \in P(h)$
- A strategy profile $(\sigma_1, ..., \sigma_N)$,
  - uniquely determines the terminal history and the outcome.

Extensive-form games

Nash equilibrium

Definition

A Nash equilibrium is a profile of strategies $(\sigma_1, ..., \sigma_N)$ such that each player's strategy is a best response to the other strategies.

Definition

A Subgame Perfect Equilibrium (SPE) is a profile of strategies $(\sigma_1, ..., \sigma_N)$ such that each player's strategy is a best response to the other strategies **in each subgame**.

Extensive-form games

Sequential entry

Example

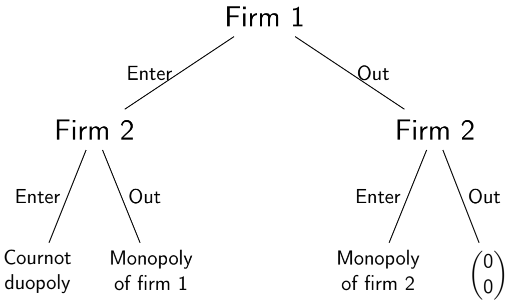
Two firms consecutively decide whether to enter the market. Firm 1 (incumbent) goes first, firm 2 (challenger) observes the decision of firm 2 and follows. The entry costs $f > 0$. After the entry, each firm chooses quantity and receives payoffs

$$\pi_i\left(q_i, q_{-i}\right) = q_i\left(a - c - \left(q_i + q_{-i}\right)\right).$$

Here, $c > 0$ is a constant marginal cost and $a > c$ is a demand parameter. The quantity of the firm which did not enter is taken as $q_{-i} = 0$.

Extensive-form games

Sequential entry



- First, we find SPE payoffs in the market game.
  - Cournot duopoly: actions

$$q_1^C = q_2^C = \frac{1}{3}(a-c),$$

profits

$$\pi_1^* = \pi_2^* = \frac{1}{9}(a-c)^2 - f.$$

  - Firm i monopoly: $q_i^{\mathrm{m}} = \frac{1}{2}(a-c)$, and profits

$$\pi^m = \frac{1}{4}(a-c)^2 - f.$$

Extensive-form games

Sequential entry: SPE

- To summarize, the SPE payoffs after each of the players makes a decision are given in the table

Firm 1\ Firm 2	E	O
E	$\frac{1}{9}(a-c)^2 - f, \frac{1}{9}(a-c)^2 - f$	$\frac{1}{4}(a-c)^2 - f, 0$
O	$0, \frac{1}{4}(a-c)^2 - f$	$0, 0$

- Histories
  - period 0: $\emptyset$,
  - period 1: E_1, O_1,
  - period 2: $E_1E_2, E_1O_2, O_1E_2, O_1O_2$

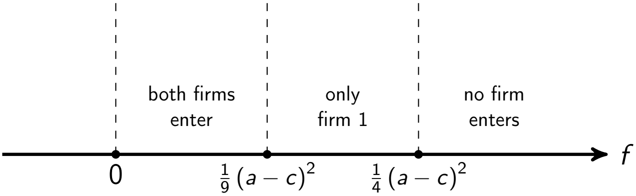
Extensive-form games

Sequential entry:SPE

Firm 1\ Firm 2	E	O
E	$\frac{1}{9}(a-c)^2 - f, \frac{1}{9}(a-c)^2 - f$	$\frac{1}{4}(a-c)^2 - f, 0$
O	$0, \frac{1}{4}(a-c)^2 - f$	$0, 0$

- After $h = E_1$
  - firm 2 enters only if $f < \frac{1}{9}(a-c)^2$,
  - firm 2 stays out if $f > \frac{1}{9}(a-c)^2$.
- After $h = O_1$
  - firm 2 enters if $f < \frac{1}{4}(a-c)^2$,
  - firm 2 stays out if $f > \frac{1}{4}(a-c)^2$.
- Best response for firm 1 is to enter if $f < \frac{1}{4}(a-c)^2$.

- Outcomes



- First-mover advantage.

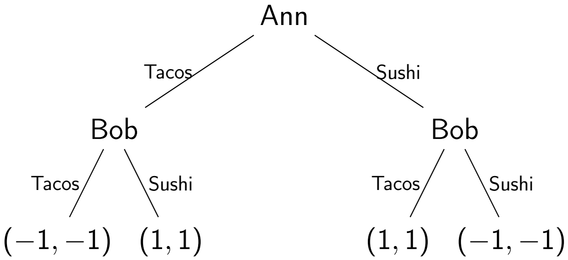
Example

Ann and Bob like to go to restaurants with their friends. There are two equally good restaurants in town. Ann likes to go out early. Bob goes out late in the evening, observes where Ann went and makes his own choice. Each one of them values each restaurant equally, but they would rather eat in an empty restaurant than in a crowded one. Payoffs

Ann\Bob	Tacos	Sushi
Tacos	-1,-1	1,1
Sushi	1,1	-1,-1

Extensive-form games

Restaurants



- There are exactly two SPE (and exactly two Nash). In both equilibria, Bob chooses to go to the other restaurant than Ann.

Plan

Introduction

Extensive-form games

Extensive form game with incomplete information

Perfect Bayesian equilibrium

Off-path beliefs

Conclusions

Extensive-form games with incomplete information

- Incomplete information can be added in many ways
  - players may learn something before the game starts,
    - in this class, mostly this,
  - they may learn something during the game,
  - players may imperfectly observe each other actions, etc.
- Incomplete information about the world
  - states of the world $\omega \in \Omega$,
  - types t_i,
  - prior belief μ over (t_i, t_{-i}, ω),
  - interim beliefs.

Extensive-form games with incomplete information

- Histories h.
  - lists of actions,
  - terminal and subhistories,
  - but also information!
- Two types
  - public *histories*: only public information
  - private *histories*: all, including private, information (types, signals, etc.)

Extensive-form games with incomplete information

- Strategies map *private histories* into actions

$$\sigma_i : \{h : i \in P(h)\} \rightarrow A_i$$

  - $\sigma_i(t_i, h)$ - action of player i type t_i after history h.
  - strategy profiles $\sigma = (\sigma_1, ..., \sigma_N)$.

Extensive-form games with incomplete information

Beliefs

- Players have beliefs $P_i\left(\omega, t_{-i} | t_i, h, \sigma\right)$
  - $P_i\left(\omega, t_{-i} | t, h\right)$ - probability of $\left(\omega, t_{-i}\right)$ as seen by player i type *after* history h,
  - beliefs may change after some histories,
  - the change may depend on the other player's actions.
- Initial beliefs are derived from Bayes formula

$$P\left(\omega, t_{-i} | t_i, \emptyset\right) = \mu_i\left(\omega, t_{-i} | t_i\right) = \frac{\mu\left(\omega, t_{-i} | t_i\right)}{\mu\left(t_i\right)}.$$

- Beliefs after on-path histories are also derived through Bayes formula.

Example

Incomplete information

- Suppose that Ann has a private information about the quality of Tacos and the quality can be either $q_H = 3$ or low $q_L = -3$.
- Bob does not know the quality and thinks that it is high with probability $p = \frac{1}{2}$.

Ann\Bob	Tacos	Sushi
Tacos	$-1 + \frac{1}{2}q, -1 + q$	$1 + \frac{1}{2}q, 1$
Sushi	$1, 1 + q$	$-1, -1$

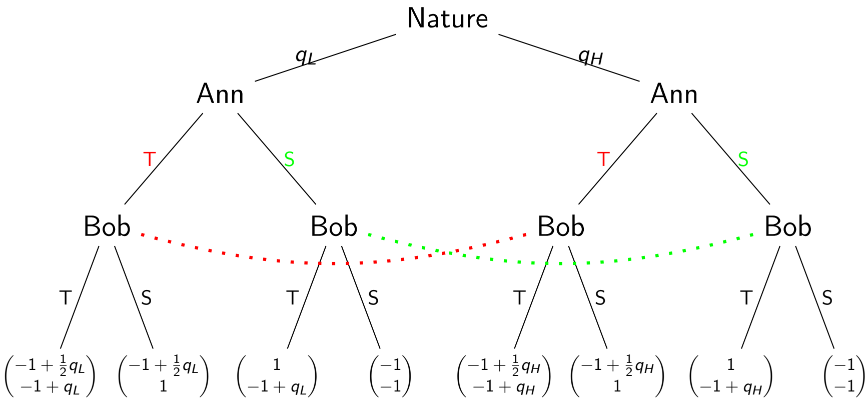
- Bob cares about quality twice as much as Ann.
- If Bob does not learn anything,

$$Eq = pq_H + (1 - p)\, q_L = 0$$

and he is indifferent between the two restaurants.

Extensive-form games

Restaurants



Extensive-form games

Restaurants: Histories

- Private Ann's histories:
  - period 1: h, l,
  - no more decision points,
- Bob's histories:
  - period 2: T_A, S_A,

Extensive-form games

Restaurants: Ann's strategies

- Ann's pure strategies: $a : \{h, l\} \to \{T_A, S_A\}$
  - follow her knowledge: $a(h) = T, a(l) = S,$
  - contrarian $a(h) = S$, $a(l) = T$,
  - always rest. T $a(h) = a(l) = T$,
  - always rest. S $a(h) = a(l) = S$,

Extensive-form games

Restaurants: Bob's strategies

- Bob's pure strategies: $b : \{T_A, S_A\} \to \{T_B, S_B\}$
  - always go to the same restaurant $b(T) = T$, $b(S) = S$,
  - always go to the other restaurant $b(T) = S, b(S) = T$,
  - always rest. T $b(T) = b(S) = T$,
  - always rest. S $b(T) = b(S) = S$.

Extensive-form games

Restaurants: Bob's beliefs

- Bob's beliefs about $\omega \in \{h, l\}$ given history $h = T_A, S_A$,
  - $P(h|T_A)$,
  - $P(h|S_A)$,
  - to compute them, we need to know what is Ann's strategy,
  - hence, we can do it only in equilibrium!

Plan

Introduction

Extensive-form games

Extensive form game with incomplete information

Perfect Bayesian equilibrium

Off-path beliefs

Conclusions

- Equilibrium
  - strategies - best responses,
  - beliefs - Bayes formula whenever possible.

Definition

A weak Perfect Bayesian Equilibrium (wPBE) is a profile of strategies $(\sigma_1, .., \sigma_N)$ and beliefs $(P_1, ..., P_n)$ such that

1. each player's strategy is a best response to other strategies given the beliefs,
2. beliefs are updated using Bayes formula, whenever possible.

Perfect Bayesian equilibrium

Pooling and separating equilibria

- If different types in equilibrium play different strategies, we say that the equilibrium is *separating:*
  - private information is going to get revealed through strategies.
- If different types play the same strategy, we say that the equilibrium is *pooling:*
  - private information is not going to be revealed.

Perfect Bayesian equilibrium

Restaurants

Lemma

There is an informative equilibrium, where Ann follows her knowledge and Bob goes to the same restaurant.

Perfect Bayesian equilibrium

Restaurants

- Ann *follows her knowledge* $a(h) = T, a(l) = S$,
- Then, Bob knows which restaurant is better (his beliefs)
  - if Ann goes to restaurant T, then T is a best response for Bob:
    - T:$-1+3=\underline{2}$, vs
    - S: 1.
  - if Ann goes to restaurant S, then S is a best response for Bob:
    - T:$1-3=-2$,
    - S: $\underline{-1}$.
- Bob's best response to "Ann following her knowledge" is "to go the same restaurant".

Perfect Bayesian equilibrium

Restaurants

- Suppose that Bob goes to the same restaurant $b(T) = T, b(S) = S$,
- Ann's payoffs
  - in state h, T is a better response:
    - T:$-1 + \frac{1}{2}3 = 0.5$, vs
    - S: -1.
  - in state l, S is a better response:
    - T:$-1 - \frac{1}{2}3 = -2.5$,
    - S: $\underline{-1}$.
- Ann's best response is to follow her knowledge.
- Separating equilibrium!
  - Ann follows her knowledge, Bob follows Ann, and Bob learns all Ann's information.

Lemma

There is an equilibrium, where Ann always goes to restaurant S. There is another equilibrium, where she always goes to restaurant T.

- Pooling equilibria.

Perfect Bayesian equilibrium

Restaurants

- Ann always goes to restaurant S, $a(h) = a(l) = S$.
- Then, when Bob observes Ann in restaurant S, Bob's beliefs are
$$P(h|S_A) = \frac{p}{p + (1-p)} = p = \frac{1}{2}.$$
  - Bob's best response is to go to restaurant T.
- What should Bob do if Ann were to go to restaurant T?

Perfect Bayesian equilibrium

Restaurants

- If Bob were to observe Ann going to restaurant T, Bob's beliefs are $P(h|T_A)$.
  - off-path history,
  - not determined by the Bayes formula.
- If Bob were to observe T_A, Bob's best response is
  - T_B if
    $$P(h|T_A)(-1+3) + (1 - P(h|T_A))(-1-3) \geq 1, \text{ or}$$
    $$P(h|T_A) \geq \frac{5}{6}.$$
  - S_B if $P(h|T_A) \leq \frac{5}{6}$.

Perfect Bayesian equilibrium

Restaurants

- Bob goes to T_B after S_A.
- If $P(h|T_A) < \frac{5}{6}$, Bob wants to go S_B after T_A
  - Bob's best response is to always go to a different restaurant than Ann.
  - Ann's best response is to always follow her knowledge.
  - Hence, this cannot be an equilibrium.
- If $P(h|T_A) \geq \frac{5}{6}$, Bob wants to go T_B after T_A
  - Bob's best response is to always go to restaurant T.
  - Ann's strategy to always go to restaurant S is a best response. We found an equilibrium!

Perfect Bayesian equilibrium

Restaurants

- Equilibrium
  - Ann's strategy: always goes to restaurant S
  - Bob's strategy: always goes to restaurant T,
  - Bob's beliefs:

$$P(h|S_A) = \frac{1}{2}, P(h|T_A) \geq \frac{5}{6}.$$

  - No need for Ann's beliefs (she knows the state of the world)

Perfect Bayesian equilibrium

Restaurants

- Convince yourself that there is also an equilibrium, in which Ann always goes to restaurant T, and Bob always goes to restaurant S.
- There is no other equilibrium (for instance, no equilibrium where Ann goes contrarian).

Perfect Bayesian equilibrium

Restaurants

- In any pooling equilibrium,
  - Ann's information is not revealed.
  - Bob never learns about the quality of the restaurant (unless he goes there, of course).

Perfect Bayesian equilibrium

Entry game with privately observed f

Example

Entry game, but with a small change.

- Prior to the entry decision, the incumbent observes the entry cost $f \in \{f_L, f_H\}$,
  - the prob. of state h is π and l is $1 - \pi$.
- We assume that

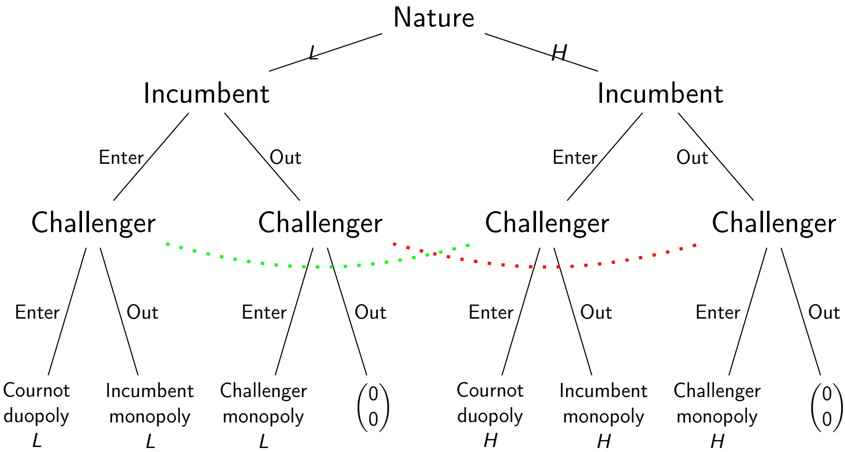
$$0 < f_L < \frac{1}{9}(a-c)^2 < f_H < \frac{1}{4}(a-c)^2,$$
$$Ef = \pi f_h + (1-\pi) f_l < \frac{1}{9}(a-c)^2$$

- The challenger does not know the true cost and observes it only if it decides to enter.

- Incumbent knows very well what is the R&D cost for a new product.
- The challenger does not know and learns it only after their

Perfect Bayesian equilibrium

Entry game with privately observed f



Perfect Bayesian equilibrium

Entry game with privately observed f

- Incumbent's strategies: $a : \{f_L, f_H\} \to \{E_I, O_I\}$
  - always enter $a(f_L) = a(f_H) = E$,
  - always out $a(f_L) = a(f_H) = O$,
  - enter when cost is high $a(f_L) = O, a(f_H) = E$,
  - enter when cost is low $a(f_L) = E, a(f_H) = O$.
- Challenger strategies $b : \{E_I, O_I\} \to \{E_C, O_C\}$.
- Beliefs:

$$P(f_H|O_I) \text{ and } P(f_H|E_I).$$

Perfect Bayesian equilibrium

Entry game with privately observed f

Lemma

The only (pure strategy) equilibrium is when the Incumbent "enters when low" and the Challenger "always enter".

- Advantage to second-mover, low-information firm!
  - compare with the "symmetric" information case.

Perfect Bayesian equilibrium

Entry game with privately observed f

- Suppose that the incumbent "always enters" $a(f_L) = a(f_H) = E_I$.
- After history $h = E_I$:
  - on-path history, beliefs $P(h|E_I) = \frac{\pi}{1} = \pi$,
  - the challenger does not learn anything from observing E.
  - the expected value of the cost is $Ef < \frac{1}{9}(a-c)^2$.
  - Challenger (expected) payoff from
    - E is $\frac{1}{9}(a-c)^2 - Ef$,
    - O is 0.
  - The challenger enters as a best response, $b(E_I) = E_C$.

Perfect Bayesian equilibrium

Entry game with privately observed f

- After history $h = O_I$:
  - off-path history, beliefs $P(h|O_I)$ are not determined in wPBE.
  - the expected value of the cost is $E(f|O_I) \in [f_l, f_h] < \frac{1}{4}(a-c)^2$.
  - Challenger (expected) payoff from
    - E is $\frac{1}{4}(a-c)^2 - E(f|O_I) > 0$,
    - O is 0.
  - The challenger enters as a best response, $b(O_I) = E_C$.

Perfect Bayesian equilibrium

Entry game with privately observed f

- Hence, Challenger's best response to "always enter_I" is "always enter_C".
- But "always enter_C" is not a best response to "always enter_I":
  - the high cost incumbent does not want to enter: payoffs from
    - E is $\frac{1}{9}(a-c)^2 - f_H < 0$, and
    - O is 0,
- So, no equilibrium, where the Incumbent "always enters".

- Next, consider a strategy "always out" $a(f_L) = a(f_H) = O_I$.
- After subhistory $h = O_I$:
  - on-path history, beliefs $P(h|E_I) = \frac{\pi}{1} = \pi$,
  - the challenger does not learn anything from observing E.
  - the expected value of the cost is $Ef < \frac{1}{9}(a-c)^2$.
  - Challenger (expected) payoff from
    - E is $\frac{1}{4}(a-c)^2 - Ef > 0$,
    - O is 0.
  - The challenger enters as a best response: $b(O_I) = E_C$

Perfect Bayesian equilibrium

Entry game with privately observed f

- After history $h = E_I$:
  - off-path history, beliefs $P(h|E_I)$ are not determined in wPBE.
  - the expected value of the cost is $E(f|E_I) \in [f_l, f_h] < \frac{1}{4}(a-c)^2$.
  - Challenger (expected) payoff from
    - E is $\frac{1}{9}(a-c)^2 - E(f|O_I) \lesseqgtr 0$,
    - O is 0.
  - The challenger best response depends on the choice of beliefs. Two cases:
    - case (a): $b(E_I) = E_C$.
    - case (b): $b(E_I) = O_C$.

Perfect Bayesian equilibrium

Entry game with privately observed f

- Case (a): Suppose $b(E_I) = E_C$.
  - Challenger's strategy is ""always enter"
  - but then Incumbent best response is to "enter when low" (CHECK IT)
  - hence, "always stay out" is not a best response,
  - no equilibrium!
- Case (b): Suppose $b(E_I) = O_C$.
  - Challenger's strategy is "always do the opposite"
  - but then Incumbent best response is to "always enter" (CHECK IT)
  - hence, "always stay out" is not a best response,
  - no equilibrium!
- In short, no equilibrium, where the Incumbent "always stays out".

Perfect Bayesian equilibrium

Entry game with privately observed f

- Next, consider a strategy "enter when high" $a(f_L) = O, a(f_H) = E$.
- The challenger's beliefs are f_L after O and f_H after E.
- Challenger (expected) payoff after O from
  - E is $\frac{1}{4}(a-c)^2 - f_L$,
  - O is 0.
  - The challenger enters after seeing O as a best response.
- But then the low cost incumbent wants to enter: payoffs from
  - E is $\frac{1}{9}(a-c)^2 - f_L > 0$, and
  - O is 0.

Perfect Bayesian equilibrium

Entry game with privately observed f

- Finally, consider a strategy "enter when low" $a(f_L) = E, a(f_H) = O$.
- Check that there is an equilibrium, in which the challenger always enters.
- Beliefs $p(a) = P(\omega = H|a)$ after choice $a_I \in \{E_I, O_I\}$
  - $p(E_I) = 0$,
  - $p(O_I) = 1$,
  - both beliefs determined in equilibrium by Bayes formula.

Perfect Bayesian equilibrium

Entry game with privately observed f

- With complete information, it was never the case that the incumbent stayed out and the challenger entered.
  - the first mover advantage.
- Incomplete information may overturn it. For some parameters, there is an equilibrium, where, sometimes, the challenger enters and the incumbent stays out.
  - inability to credibly communicate the true cost parameter hurts the incumbent.
  - extra information can be bad.

Plan

Introduction

Extensive-form games

Extensive form game with incomplete information

Perfect Bayesian equilibrium

Off-path beliefs

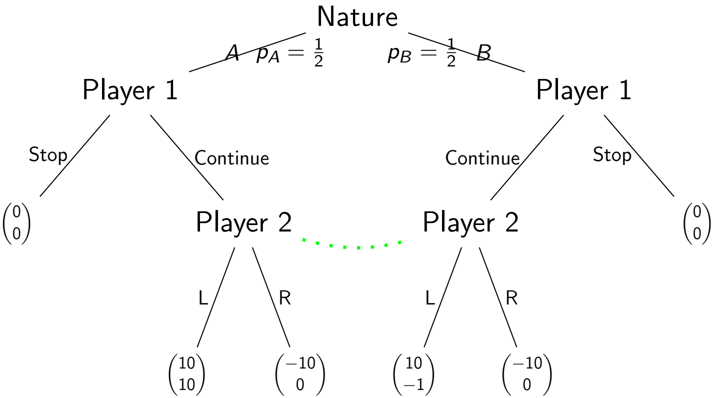
Conclusions

Off-path beliefs

- A part of the description of the equilibrium are beliefs.
  - we need them to check whether strategies are best responses,
  - both on- and off-path.
- On-path - derived from Bayes formula.
- Off-path - arbitrary, at least in WPBE.
  - Not determined by strategies, arbitrary.
  - often, too permissive.

Off-path beliefs

- Example.



Off-path beliefs

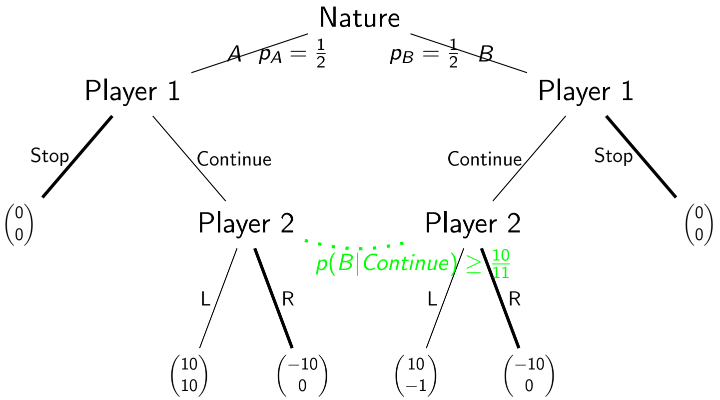
- This game has an equilibrium in which player 1 plays Stop.
- For such an equilibrium, it is necessary that after C, Player 2 chooses R
  - Payoff from L:

$$\begin{aligned} & 10p_2\left(A|C\right)+\left(-1\right)p_2\left(B|C\right) \\ = & 10\left(1-p_2\left(B|C\right)\right)-p_2\left(B|C\right) \\ = & 10-11p_2\left(B|C\right) \end{aligned}$$

  - Payoff from R: is 0.
  - For R to be best response, player 2 must believe that the state is B with probability
$$p_2\left(B|C\right)\geq\frac{10}{11}.$$
  - That's possibly because Player 1 plays Contniue only because she made a mistake and maybe, she makes more mistakes in state B.

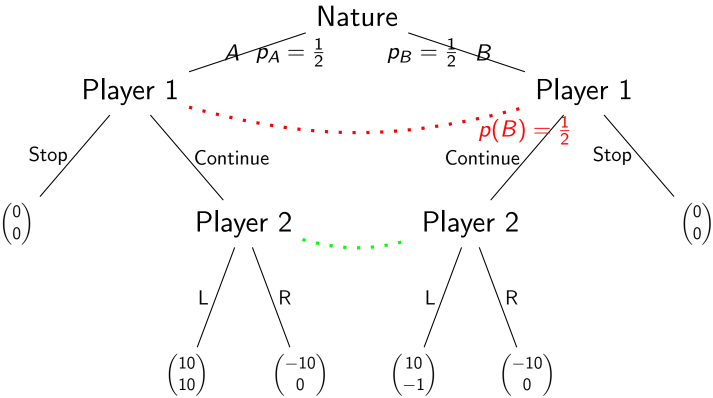
Off-path beliefs

- wPBE
  - Player 1 plays Stop in each state.
  - Player 2 plays R with beliefs $p\left(B|C\right) \geq \frac{10}{11}$.



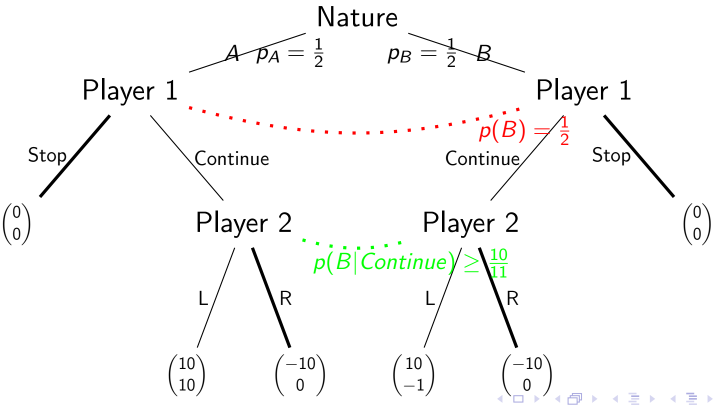
Off-path beliefs

- But what if player 1 does not know the state?



Off-path beliefs

- Still, wPBE
  - Player 1 plays Stop with beliefs $p(B) = \frac{1}{2}$.
  - Player 2 plays R with beliefs $p(B|\text{Continue}) \geq \frac{10}{11}$ (off-path, no Bayes formula).
  - Inconsistency between the beliefs of player 2 and the fact that player 1 does not know the state!



Off-path beliefs

- In the above example, the restriction that the beliefs of player 1 and 2 are equal seems natural.
  - player 1 does not know the state of the world, so no reason why his beliefs are different.
- But how to formalize a “natural” restriction for a general game?

Off-path beliefs

- Game Theory came up with many ideas for "natural" restrictions for off-path beliefs:
  - Perfect Bayesian Equilibrium,
  - sequential equilibrium,
  - proper equilibrium,
  - intuitive criterion,
  - divine equilibrium,
  - D1,D2 criterion
  - refinements in communication games
  - etc.

Off-path beliefs

- Sequential equilibrium: strategies σ and beliefs P st.
  - σ, P are a wPBE,
  - there is a sequence of full support strategies $\sigma_k \to \sigma$ such that
    - if P_k are beliefs derived from Bayes formula (possible everywhere, given full support of σ_k),
    - then $P_k \to P$
- The equilibrium in the last example is not sequential.

Plan

Introduction

Extensive-form games

Extensive form game with incomplete information

Perfect Bayesian equilibrium

Off-path beliefs

Conclusions

Conclusions

What did we learn - concepts

- Extensive form games.
- Nash and Subgame perfect equilibrium.
- Extensive form games with incomplete information.
- Weak Perfect Bayesian Equilibrium.

Conclusions

What did we learn - skills

- Find equilibria of extensive form games.
- Use Bayes formula and players' strategies to update beliefs.
- Find equilibria of simple extensive form games with incomplete information.
- Check whether there is an equilibrium with a particular property.