ECO421: Knowledge

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Plan

Introduction

- This class is an continuation of game theory classes from 3rd year.
  - game theory is about strategic behavior,
  - special case: games with incomplete information.
- We focus on games with incomplete information.
- We want to know how incomplete and asymmetric information affects how people behave.

Introduction

- Grading:
  - midterm (30%)
  - final (30%)
  - writing assignment (40%)
- Midterm and Final are based on the examples from lecture notes (slides).
- Final is (almost) not cumulative.

Introduction

- Writing assignment: Find a real-world example of strategic situation with incomplete information, build a model, and analyze it.
- Four parts:
  - informal description (cannot be made up, sources),
  - model (formal description of the game),
  - results (equilibrium, maybe some analysis),
  - conclusions (what did we learn, how we can use the results).

Plan

Describing information

- Our first task is foundational. We want to design a way to talk about information, and differences in information.
- We want a model that is rich enough to
  - describe differences in information between people,
  - allow for learning new information.
- Two steps:
  - (today) knowledge, knowledge about knowledge, etc.
  - (next class) beliefs.

Describing information

- Information should tell us something about what the world really is, but not necessarily everything.
- *(A piece of) information*: an answer to some questions.
- *Information structure*: the complete set of questions answers to which one expects to get.
- *State of the world*: complete description of the world, including answers to questions that we are not going to learn.

Describing information

Example

Thermometer:
I have a thermometer that allows me to measure the temperature outside with the precision of 1 degree C.
More precisely, if the temperature is in $\tau \in [m - 0.5, m + 0.5)$ for some integer $m \in Z$, the thermometer is going to show m.

- The thermometer defines the information structure.
- A particular measurement m is a (piece of) information.
- A state of the world is the exact value of temperature τ.

Describing information

- We will think about the information structure in a more abstract way.
- Three concepts
  - a state of the world:
  - a type (i.e., a piece) of information,
  - information structure.

Describing information

States of the world

- *State of the world* $\omega \in \Omega$
  - all relevant description of the world,
  - Ω is the space of all states of the world.

Describing information

Types (of information):

Definition

A set $E \subseteq \Omega$ is called an *event* or a *piece* or a *type of information.*

- If the agent knows information E, then she or he knows that
  - the *true state of the world* belongs to E,
  - states in $E^c = \Omega \backslash E$ (i.e., the complement of E) are not true
  - but she considers all states in E as possible:
    - she does not know which of the states in E is the true one (unless E contains only one state)

- Operations on pieces of information, $E \subseteq \Omega$ and $F \subseteq \Omega$:
  - "E and F" corresponds to $E \cap F$,
  - "E or F" corresponds to $E \cup F$,
  - "not E" corresponds to $\Omega \backslash E$,

Describing information

Definition

Information structure $\mathcal{T}_i$ of agent i is a partition of Ω into types of information.

- Informally, the information structure is the set of all different answers that the agent can receive.
- An answer determines the set of states (i.e., piece of information) that are consistent with this particular answer.
- For each ω, let $T_i(\omega) \in \mathcal{T}_i$ be the answer (i.e., the information) that agent has in state ω.

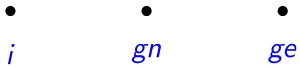
Example

D. Trump knows whether he colluded with Russia or not. R. Mueller knows whether he found an evidence of collusion. The evidence exists only if Trump is guilty. None of them knows anything else.

Describing information

States of the world: Investigation

- In the Investigation example, there are 3 states of the world:
  - Trump is innocent (i),
  - Trump is guilty, but Mueller has no evidence (gn),
  - Trump is guilty, and Mueller has the evidence (ge).



Describing information

Types (of information): Investigation

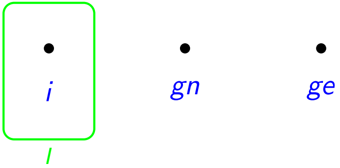
- Trump knows whether he is guilty, or innocent.

i $\qquad gn$ $\qquad ge$

Describing information

Types (of information):Investigation

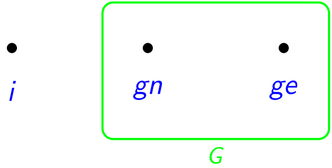
- Trump can either know that he is innocent, $I = \{i\}$,



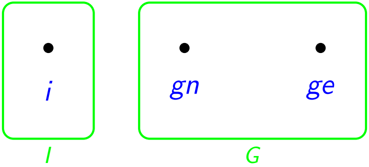
Describing information

Information structure: Investigation

- Or, Trump can know that he is guilty, $G = \{gn, ge\}$,

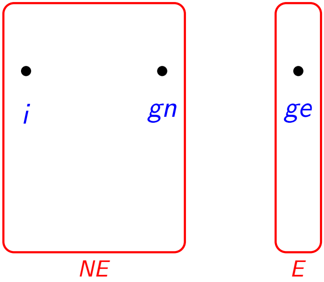


- Trump's informational structure:



Mueller's informational structure: He knows whether

- there is evidence, $E = \{ge\}$,
- or not, $NE = \{i, gn\}$.



Describing information

Type space

Definition

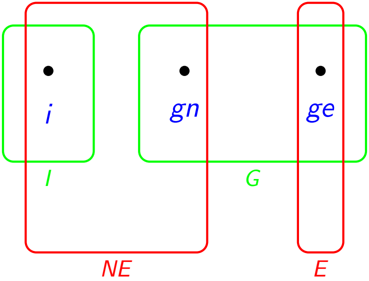
A (knowledge-based) type space is a collection of information structures for each player:

$$(\Omega, (\mathcal{T}_i)) .$$

- Each player has an information structure.
- Information structures for each player form a type space.

Describing information

Type space in Investigation



Plan

Knowledge

- Let $E \subseteq \Omega$ be a piece of information.
- What does it mean that an agent knows that E is true?
- Here we take :"know E" means that "the agent is certain that E is true".
- "Knowledge" event depends on the state (as her information depends on the state):
  - in some states, the agent may know E,
  - sometimes she may be not sure whether E is true or not,
  - sometimes, she may be sure that E is not true.
- In the last two cases, we say that she "doesn't know" E.

Knowledge

Definition

An agent i knows E in state ω ("knows that E is true in state ω") if $T_i(\omega) \subseteq E$.

- $T_i(\omega)$ is the set of all states that i considers possible given her information in state ω.
- i knows E if she is sure that E is true, i.e., if all possible state belong to E.

Knowledge

Definition

The set of states where i knows E is

$$\begin{aligned} K_i(E) &= \{\omega : i \text{ knows } E \text{ in state } \omega\} \\ &= \{\omega : T_i(\omega) \subseteq E\}. \end{aligned}$$

Knowledge

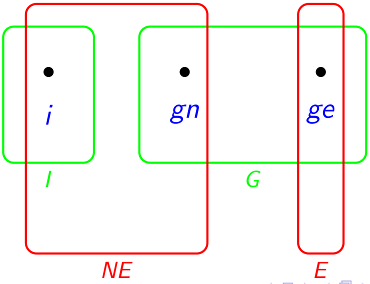
Knowledge: Investigation

- Mueller wants to know whether Trump is guilty or Innocent.
- In state ge, Mueller knows that

$$T_{\text{Mueller}}(ge) = \{ge\} \subseteq G.$$

- In states i, gn, Mueller knows neither G nor I:

$$T_{\text{Mueller}}(i) = T_{\text{Mueller}}(gn) = \{i, gn\} \nsubseteq G,$$
$$T_{\text{Mueller}}(i) = T_{\text{Mueller}}(gn) = \{i, gn\} \nsubseteq I.$$



Knowledge

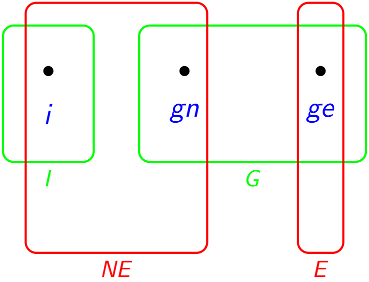
Knowledge: Investigation

- Mueller knows that Trump is guilty only if he has the evidence,

$$K_{\mathrm{Mueller}} G = \{ge\},$$

- Mueller never knows (for sure) that Trump is innocent,

$$K_{\mathrm{Mueller}} I = \emptyset.$$

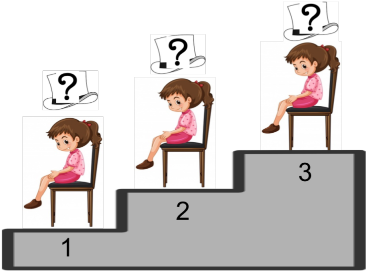


Plan

Example: Hats

Example

Three girls sit on chairs, in a row.
Each girl has a hat on her head. The hats are either black or white.
Girl #3 sees girls #1 and #2 and girl #2 sees girl #1. None of them can see her own hat.

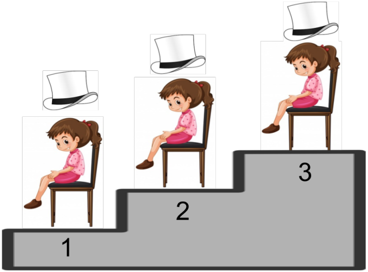


Example: Hats

Example

Suppose that each girl has a white hat.

- The girls are asked to guess the color of their hats. Silence follows.
- Next, the girls are informed that there are AT MOST 2 black hats. After some time, the girl #1 exclaims: "I have a white hat!"
- What is her reasoning?



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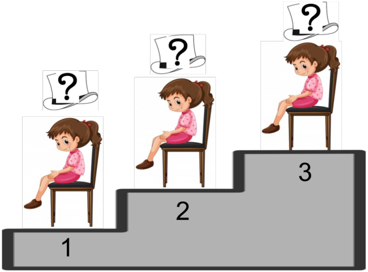


Example: Hats

- In the Hats example, there are eight states of the world:

$$\Omega = \{1, 2, 3, 4, 5, 6, 7, 8\} .$$

	1	2	3	4	5	6	7	8
Hat of girl 1	●	●	●	●	○	○	○	○
Hat of girl 2	●	●	○	○	●	●	○	○
Hat of girl 3	●	○	●	○	●	○	●	○

Example: Hats

Information structure: Hats, Girl #1

Girl #1 has no information,

- she cannot distinguish any two states of the world,
- her information structure has only one type $T = \{\Omega\}$.

	1	2	3	4	5	6	7	8
Hat of girl 1	●	●	●	●	○	○	○	○
Hat of girl 2	●	●	○	○	●	●	○	○
Hat of girl 3	●	○	●	○	●	○	●	○

Ω

Example: Hats

Information structure: Hats, Girl #2

Girl #2 can have one of two types of information:

- the hat of Girl #1 is black $B_1 = \{1, 2, 3, 4\}$, or
- the hat of Girl #1 is white $W_1 = \{5, 6, 7, 8\}$.i

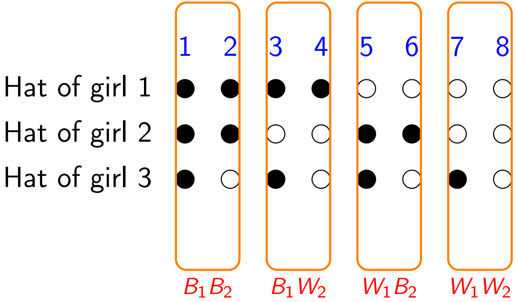
	1	2	3	4	5	6	7	8
Hat of girl 1	●	●	●	●	○	○	○	○
Hat of girl 2	●	●	○	○	●	●	○	○
Hat of girl 3	●	○	●	○	●	○	●	○
	B_1				W_1			

Example: Hats

Single agent information: Hats, Girl #3

Girl #3 can have one of four types of information:

- Girl #1's hat is black and Girl #2's is black, $B_1B_2 = \{1, 2\}$,
- Girl #1's hat is black and girl #2's is white, $B_1W_2 = \{3, 4\}$,
- etc.



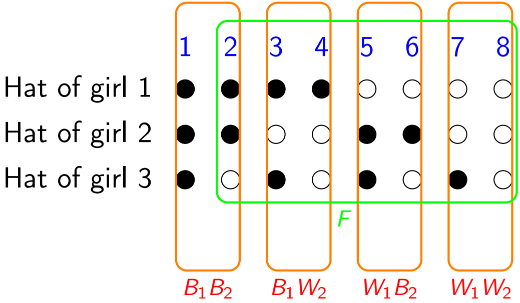
Knowledge

Knowledge: Hats

- When does Girl #3 know that

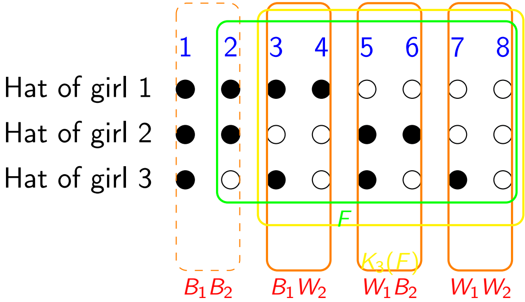
$$F = \text{"there are at most 2 black hats"?}$$

Notice that $F = \{2, 3, ..., 8\}$.



- When does Girl #3 know that

$$F = \text{"there are at most 2 black hats"?}$$
$$= \{2, 3, ..., 8\}?$$



$$K_3(F) = \{3, 4, ..., 8\}$$

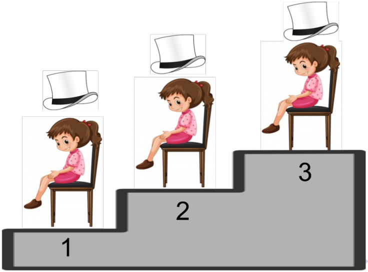
Reasoning about knowledge

Example: Hats

Example

Suppose that each girl has a white hat.

- The girls are asked to guess the color of their hats. Silence follows.
- Next, the girls are informed that there are AT MOST 2 black hats. After some time, the girl #1 exclaims: "I have a white hat!"
- What is her reasoning?



Reasoning about knowledge

Updating from new information

- How do we describe learning, or updating given new information?
- Old information structure $(\Omega, (T_i(.)))$.
- New piece of information $F \subseteq \Omega$: "the true state is in F"
- New information structure $\left(\Omega^F, \left(T_i^F(.)\right)\right)$.
  - new state space $\Omega^F = \Omega \cap F$,
  - new types $T_i^F = T_i(\omega) \cap F$ for each $\omega \in \Omega^F$.

- New information structure $\left(\Omega^F, \left(T_i^F(.)\right)\right)$.
  - new state space $\Omega^F = \Omega \cap F$,
  - new types $T_i^F = T_i(\omega) \cap F$ for each $\omega \in \Omega^F$.

Definition

The set of states where E is known given F is

$$K_i(E|F) = \left\{\omega : T_i^F(\omega) \subseteq E\right\}$$
$$= \{\omega : T_i(\omega) \cap F \subseteq E\}.$$

Reasoning about knowledge

Example: Hats

- The reasoning goes through three steps.
- Step 1: All girls learn F.
- Step 2: After initial period of silence, all girls learn that, additionally to F, girl #3 does not know her hat. Hence, they learn

$$\begin{aligned} F' &= F \text{ and "girl \#3 knows } F \text{ but she does not know her hat"} \\ &= F \text{ and not } \left(K_3\left(W_3|F\right) \text{ or } K_3\left(B_3|F\right)\right) \end{aligned}$$

- Step 3: After initial period of silence, all girls learn that, additionally to F', girl #2 does not know her hat. Hence, they learn

$$\begin{aligned} F'' &= F' \text{ and "girl \#2 knows } F' \text{ but she does not know her hat"} \\ &= F' \text{ and not } \left(K_2\left(W_2|F'\right) \text{ or } K_2\left(B_2|F'\right)\right) \end{aligned}$$

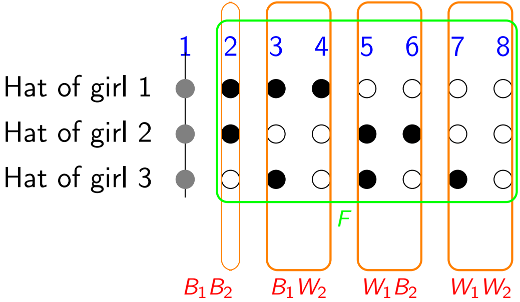
- Step 3: Upon learning F'', girl #1 concludes that she has a white hat.

Reasoning about knowledge

Example: Hats

- Step 1: Girl #3 learns $F = \{\text{at most 2 black hats}\}$.

$$K_3\left(W_3|F\right) = \{2\}\,,$$
$$K_3\left(B_3|F\right) = \emptyset.$$



Reasoning about knowledge

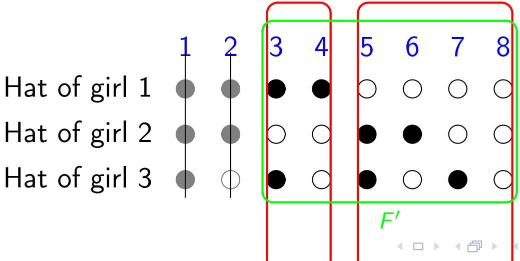
Example: Hats

- Step 2. Girl #2 keeps quiet, hence girl t#2 learns that neither $K_3\left(W_3|F\right)$ nor $K_3\left(B_3|F\right)$ are true.
  Girl #2 learns

  $$\begin{aligned} F' &= F \text{ and not } \left(K_3\left(W_3|F\right) \text{ or } K_3\left(B_3|F\right)\right) \\ &= F \backslash \left(K_3\left(W_3|F\right) \cup K_3\left(B_3|F\right)\right). \end{aligned}$$

  Then,

  $$\begin{aligned} K_2\left(W_2|F'\right) &= \{3,4\}, \\ K_2\left(B_1|F\right) &= \emptyset. \end{aligned}$$



Reasoning about knowledge

Example: Hats

- Step 3. Girl #2 keeps quiet, hence girl #1 learns that neither $K_2\left(W_2|F'\right)$ nor $K_2\left(B_2|F'\right)$ are true (additionally to what she learned in step 2, i.e., F'').

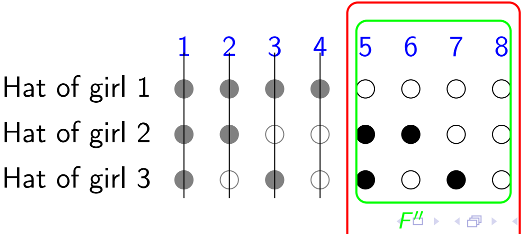
  Girl #1 learns

$$\begin{aligned} F'' &= F' \text{ and not } \left(K_2\left(W_2|F'\right) \text{ or } K_2\left(B_2|F'\right)\right) \\ &= F' \setminus \left(K_2\left(W_2|F'\right) \cup K_2\left(B_2|F'\right)\right) = \{5,6,7,8\}\,. \end{aligned}$$

  Then,

$$K_1\left(W_1|F''\right) = \{5,6,7,8\} = F''.$$

  Hence, girl #1 learns that her hat is white!



Plan

Knowledge hierarchies

- We can also talk about what one player knows that the other player knows, etc.
- For instance, the set of states in which player 2 knows that player 1 knows that event E is true:

$$K_2\left(K_1\left(E\right)\right)$$

Knowledge hierarchies

Applications

- Key element of strategic reasoning.
- Evidence of theory of mind, i.e., the ability to attribute mental states to others (here, cognitive states)
  - one of the key differences between animals and humans
    - there is some evidence that some animals (chimpanzees, dogs) do have theory of mind.
  - one of the earliest effects of socialization,
    - autism as a deficit of theory of mind.
- Deception: spectacular cons.
- Common knowledge (later)
- Funny conversations in comedies: http://tvtropes.org/pmwiki/pmwiki.php/Main/IKnowYouKnowIKnow
  - see also TV tropes.

Knowledge hierarchies

Example: Two generals

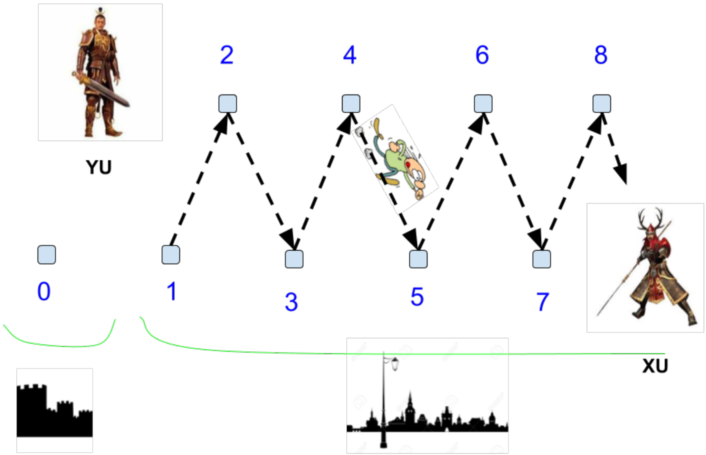
Example

Two allied generals approach an enemy city from two different sides. In order to be successful, the two generals must attack simultaneously and only if the city's fortifications are weak. Only General Xu is in position to observe the fortifications.

- If Xu observes that successful attack is possible, he sends a messenger to General Yu with this information. On its way, the messenger faces the risk of being captured by the enemy.
- If the messenger arrives safely, General Yu immediately sends him back to let General Xu know that she got the message.
- In turn, if the messenger reaches the camp of General Xu, he is immediately sent again to General Yu to inform Yu that Xu got the message.
- The messenger continues running back and forth until he gets caught.

Knowledge hierarchies

Example: Two generals



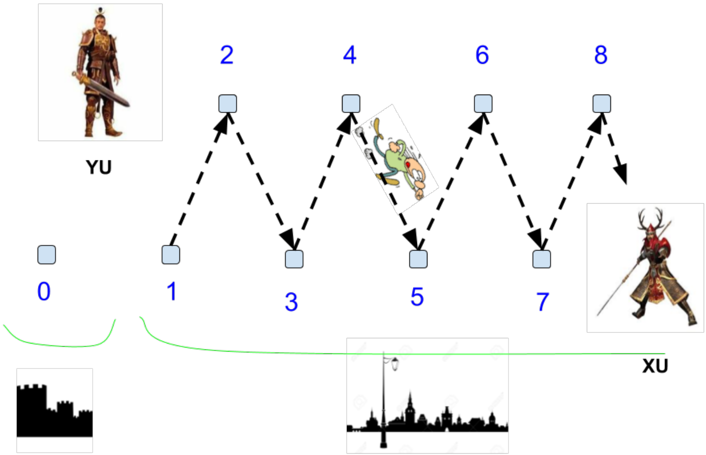
Knowledge hierarchies

Example: Two generals

- The world is described by the number of times the messenger was sent away $\omega \in \Omega = \{0, 1, 2, 3, ...\}$.
- Two important events:
  - city is strong: $S = \{0\}$,
  - city is weak: $W = \{1, 2, 3, 4, ...\} = \Omega \setminus S$.

Knowledge hierarchies

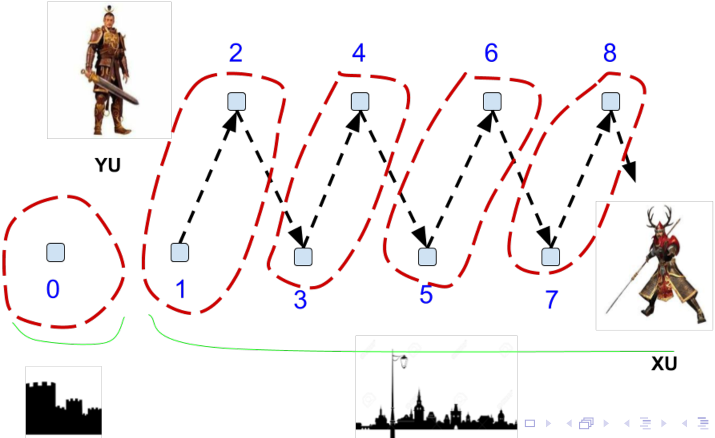
Two generals: Information structure of General Xu



Knowledge hierarchies

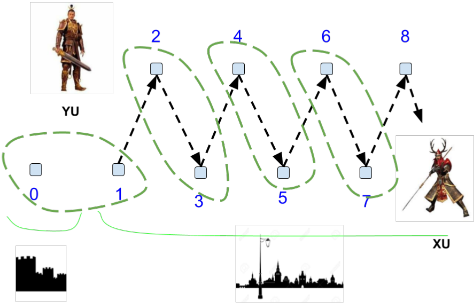
Two generals: Information structure of General Xu

- Xu cannot distinguish between state 1 and 2. In both states, he has the same information "I sent the messenger once".
- Similarly, in states 3 and 4, Xu has the same information " I sent the messenger twice".



Knowledge hierarchies

Two generals: Information structure of General Yu



Knowledge hierarchies

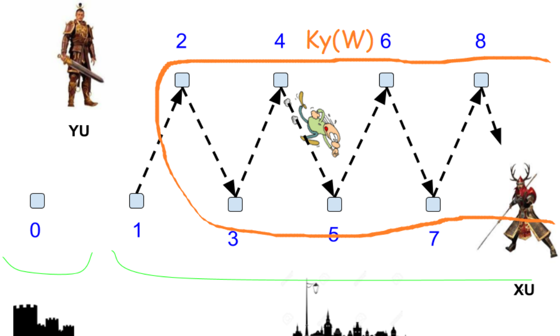
Two generals

- Both generals need to know that the city is Weak to attack. Xu observes the strength of the city:

$$K_{Xu}\left(W\right) = W = \left\{1, 2, 3, ...\right\},$$

- Yu only knows that the city is weak if he receives the messenger:

$$K_{Yu}\left(W\right) = \left\{\cancel{1}, 2, 3, ...\right\}.$$

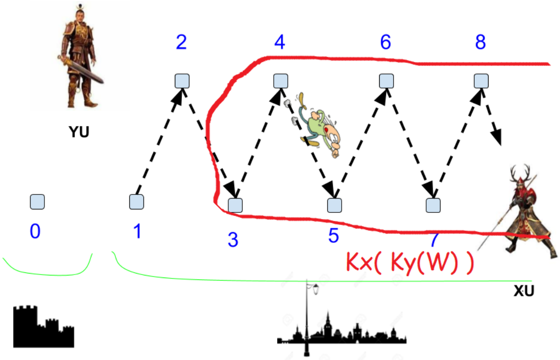


Knowledge hierarchies

Two generals

- Xu knows that Yu knows only when the messenger comes back to Xu (hence, the messenger runs at least 3 times):

$$K_{Xu}(K_{Yu}(W)) = \{\cancel{1}, \cancel{2}, 3, 4, 5, \dots\}$$

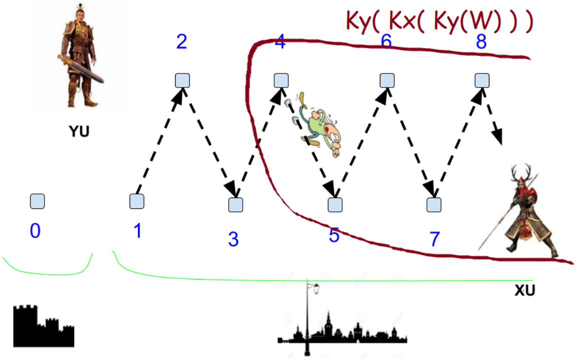


Knowledge hierarchies

Two generals

- Yu knows that Xu knows that Yu knows only when the messenger comes back to Yu for the second time (hence, the messenger runs at least 4 times):

$$K_{Yu}(K_{Xu}(K_{Yu}(W))) = \{\cancel{1, 2, 3}, 4, 5, \ldots\}$$

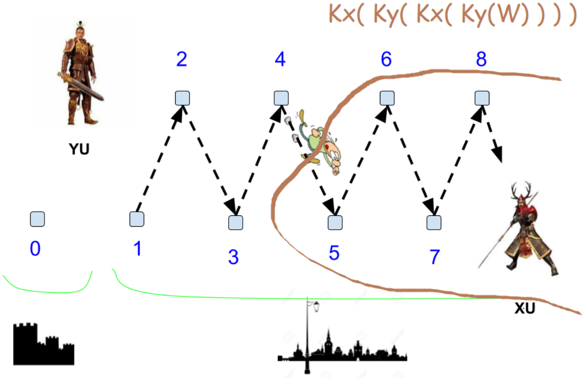


Knowledge hierarchies

Two generals

- Xu knows that Yu knows that Xu knows that Yu knows only when the messenger comes back to Xu for the third time time (hence, the messenger runs at least 5 times):

$$K_{Xu}(K_{Yu}(K_{Xu}(K_{Yu}(W)))) = \{\cancel{1, 2, 3, 4}, 5, \ldots\}$$



Knowledge hierarchies

Two generals

- In the example, attack is successful only if the two generals attack simultaneously and only if the fortifications are weak. Otherwise, the attacking general army will be destroyed.
- None of the generals wants to destroy their armies.
- None of the generals will attack unless they know that the city is weak,
  - and that the other general knows that the city is weak,
  - and that the other general knows that he knows that the city is weak,
  - etc.
- They need *common knowledge*.

Plan

Common knowledge

Definition

An event E is common knowledge in state ω if

- everybody knows that E is true at ω,
- everybody knows that (everybody knows that E is true) at ω,
- everybody knows that (everybody knows that everybody knows that E is true) at ω,
- etc.

Common knowledge

Definition

An event E is common knowledge in state ω if

- everybody knows that E is true at ω,
- etc.,

$$\omega \in \bigcap_i K_i(E)$$

$$\ldots.$$

Common knowledge

Definition

An event E is common knowledge in state ω if

- everybody knows that E is true at ω,
- everybody knows that (everybody knows that E is true) at ω,
- etc.

$$\begin{aligned}\omega \in & \bigcap_i K_i(E) \\ & \cap \bigcap_j K_j\left(\bigcap_i K_i(E)\right)\end{aligned}$$

Common knowledge

Definition

An event E is common knowledge in state ω if

- everybody knows that E is true at ω,
- everybody knows that (everybody knows that E is true) at ω,
- everybody knows that (everybody knows that everybody knows that E is true) at ω,
- etc.

$$\begin{aligned}\omega \in & \bigcap_i K_i(E) \\ & \cap \bigcap_j K_j\left(\bigcap_i K_i(E)\right) \\ & \cap \bigcap_k K_k\left(\bigcap_j K_j\left(\bigcap_i K_i(E)\right)\right) \\ & \cap \ldots\end{aligned}$$

Common knowledge

Two generals

- For everybody to want to attack, we need the common knowledge that the city is Weak.
- But in this example, there is never common knowledge.
  - Indeed, if Xu thinks that Yu is worried that the city is strong, Xu thinks that Yu won't attack. Hence, Xu won't attack.
  - If Yu thinks that Xu thinks that Yu thinks that the city is Strong, Yu thinks that Xu won't attack. Hence, Yu won't attack.
  - And so on ....
- Hence, nobody will ever attack.

Definition

Common knowledge

- The notion of "Common knowledge" was first formulated by philosopher D. Lewis who talked about language.
- Sort of independent appearance in
  - computer science, and
  - game theory.
- Huge career in 90ies.

Common knowledge

Application: Language

- Common knowledge is always important whenever there is a role for coordination.
- Example: Language.
- When I say "Please pass the salt" I assume that my wife knows that "salt" does not mean pepper.
- But I also assume that she knows that I know that "salt" does not mean pepper.
- But I also assume that she knows that I know that she knows that "salt" does not mean pepper.
- Etc.

Common knowledge

Application: Superbowl adds

- Example: Superbowl ads (M. Chwe).
- Superbowl is most watched event in US television (~130 mln viewers).
- Most expensive ads.
- But, the price of ads per viewer is higher than the price any other time
- Many advertised goods are coordination goods. Become more profitable, if they create coordination among large numbers of people.
  - "Go Daddy" website,
  - movies.

Common knowledge

Application: collective action

- Example: Protests in authoritarian state.
- Imagine that you live in a country ruled by a tyrant with powerful security force.
- You hate the guy and you are willing to take a significant risk and protest (rebel, revolt, etc.)
- The risk is much smaller if many people join you.
- Coordination game: I would like to go if many people go as well.
- Authoritarian states control mass media
- Question: How did Guy Fawkes organize protest in "V for Vendetta"?

Common knowledge

Application: coup d'etat

- E. Luttwak in "Coup d'etat" gives a recipe for a successful coup.
- One of the first thing for the rebels to do is take control over media (radio and TV).
- Two roles:
  - releases common knowledge signal: "We are strong enough to take over important installation. Don't resist us.
  - makes it impossible for the regime supporters to send a similar common knowledge signal and to coordinate their response.

Plan

Conclusions

What did we learn - concepts

- States of the world.
- Information. Information structure.
- Knowledge. Knowledge sets.
- Knowledge hierarchies.
- Common knowledge.

Conclusions

What did we learn - skills

- Represent a story with asymmetric information with state space and information of players.
- Explain how new information affects the information structure.
- Reason about knowledge and knowledge about knowledge.

Conclusions

Further reading

- Reasoning about knowledge
  - Aumann, Robert (1976) "*Agreeing to Disagree*" Annals of Statistics 4(6): 1236–1239.
  - Fagin, Ronald; Halpern, Joseph; Moses, Yoram; Vardi, Moshe (2003). *Reasoning about Knowledge*. Cambridge: MIT Press.
  - *The Electronic Mail Game: Strategic Behavior Under "Almost Common Knowledge"* Ariel Rubinstein, The American Economic Review Vol. 79, No. 3 (Jun., 1989), pp. 385-391
- Common knowledge
  - Lewis, David (1969) *Convention: A Philosophical Study* Oxford: Blackburn.
  - *Rational Ritual: Culture, Coordination, and Common Knowledge* by Michael Suk-Young Chwe , Princeton.