ECO421: Reputation

Marcin Pęski

March 15, 2020

Plan

Reputation

- This class is about a game theoretic model of reputation.
- There are many meanings of "reputation". Here, we are interested in reputation for something
  - reputation for keeping promises,
  - providing high quality service (good plumber),
  - being vengeful (always hitting back), etc.
- Common thread
  - reputation for some characteristics, or personal quality that is associated with a particular behavior,
  - in order to maintain reputation, some behavior is required,
  - reputation is valuable.

Reputation

- Reputation in game theory was discovered a bit by accident.
- Folks looked at dynamic games and noticed problems with Subgame Perfect Equilibria.
- Predictions of a standard solution, a Subgame Perfect Equilibrium, were very weird, unintuitive, and contradicted by experiments.
  - centipede game,
  - chain store game.
- A proposal to address the problem turned out to look like a model of a reputation.

Centipede game

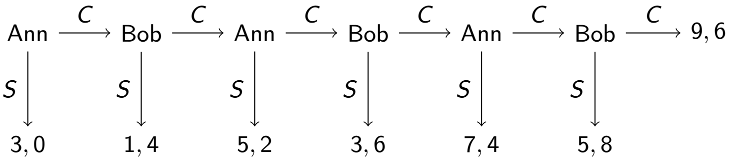
Example

Centipede game:

- Two players, Ann and Bob. There are \$3 on the table.
- The game lasts for at most $T < \infty$ periods. Each period that the game is still on, the pile on the table increases by \$2.
  - Value of the pile $= 3, 5, 7, ..., 2T + 1$.
- Players may stop in alternating periods: Ann in odd periods, Bob in even.
- If game is stopped in period t,
  - players split the pile in half,
  - the stopping player gets \$1.5 from the other player,
  - so, a player who stops in period t gets $t + 2$ and the other player gets $t - 1$.
- If nobody stops till period T, as if the game stopped n period $T + 1$.

Centipede game

- Example: $T = 6$



Centipede game

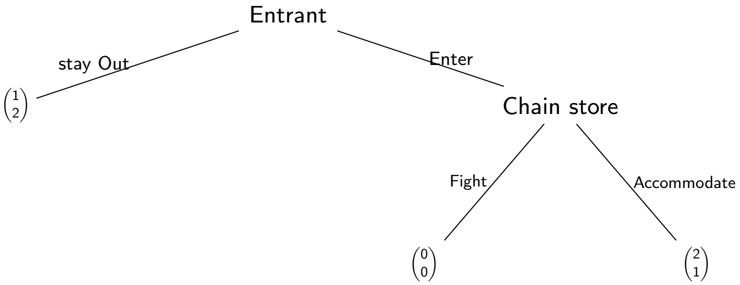
- In any centipede game, the unique SPE is to stop in the very first period.
- But that's crazy when T is large, say $T = 100$ - players get payoffs $(3, 0)$, when they could wait till the end and get ~ 100.
- Experiments do not behave as in SPE - players do not stop at the beginning and stop, at random times, close to the end.

Chain store game

- A large Chain Store company (say, Indigo) has stores in multiple, T markets.
- In each market, there is a potential Entrant (independent bookstore),
  - entrants $t = 1, ..., T$,
- The Entrants decide, one after another, whether to enter.
- If the Entrant enters, the Chain store decides whether to Fight or Accommodate.
- The subsequent entrants observe the outcomes of previous interactions.
- Chain store's payoffs are sum of payoffs from each market. Each entrant receives a payoff from the interaction in their market.

Chain store game

Stage game (one market)



- One shot game: Entrant and Chain store
- unique SPE: Enter, followed by Accommodate

Chain store game

Multiple periods: Subgame perfect equilibrium

- We start with the last period T.
- The game ends at the end of this period.
- At this moment, the game looks like one-shot.
  - the SPE behavior in subgame h is the same as in the one-shot game:
  - Enter, followed by Accommodate
- This observation does not depend on the history h before period T.

Chain store game

Subgame perfect equilibrium

- Consider period $t = T - 1$.
- The behavior in period T does not depend on what happens in period $T - 1$.
- Hence, at this moment, the game looks like one-shot.
  - the SPE behavior in subgame h is the same as in the one-shot game:
  - Enter, followed by Accommodate
- This observation does not depend on the history h before period T.
- Etc.

- Mutiple period game: Unique SPE
  - in each period, Enter, followed by Accommodate
- But?

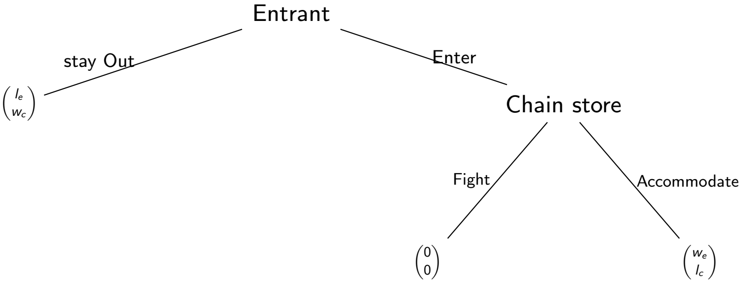
Chain store game

Subgame perfect equilibrium

- R. Selten in 70ies proposed this example as a thought experiment to illustrate the problems with
- In most experiments, the Chain Store will keep on fighting till some of the last periods,
  - after perhaps few attempts, Entrants will learn that the Chain Store fights,
  - they won't enter.
  - the outcome is that, in most markets, the Entrants stay Out
  - much better for the Chain Store.
- Problem for a game theory.
- Solution: A "small" modification of the original model
  - we add a bit of incomplete information.

Plan

Chain store with reputation



- More general payoffs: $w_i > l_i > 0$

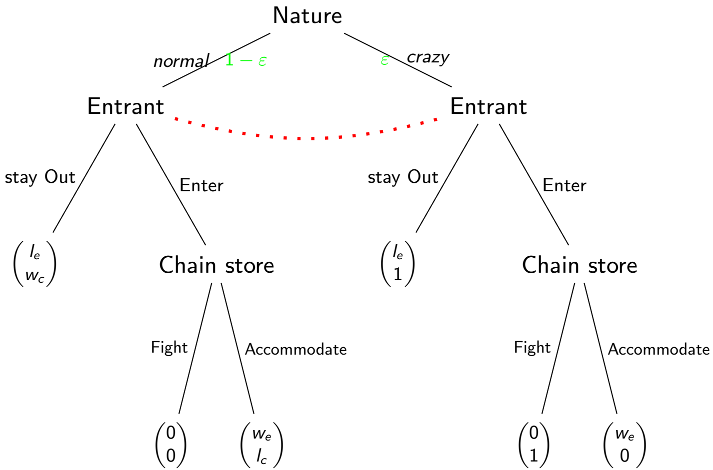
Chain store with reputation

- A "small" modification of the chain store: add small amount of incomplete information
- Two types of the Chain Store:
  - normal, with probability $1 - \varepsilon < 1$
    - the same payoffs as in the original game,
  - *crazy*, with probability $\varepsilon > 0$,
    - payoff 1 if Fights, 0 otherwise,
    - always Fights.
  - ε is very small, but strictly positive,
- If the modification is small, does it matter?

Chain store with reputation

One-shot game

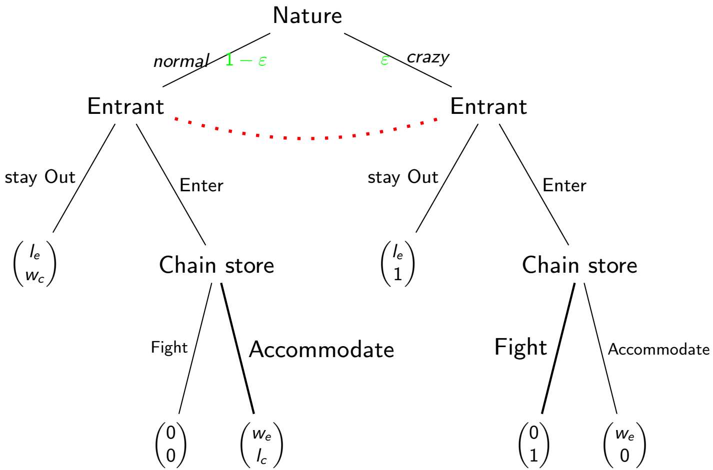
- One-shot case:



Chain store with reputation

One-shot game

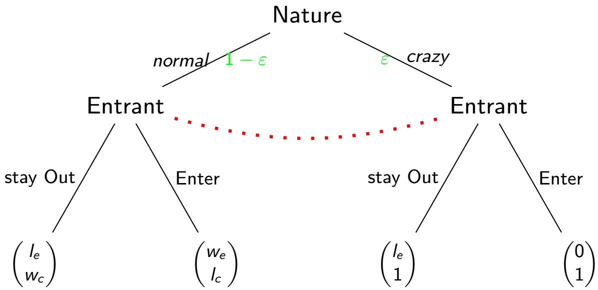
- One-shot case:



Chain store with reputation

One-shot game

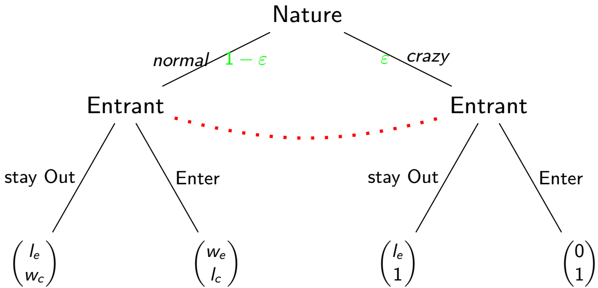
- We can replace the Chain store's decision.



- Entrant's expected payoffs from Out$\Longrightarrow$ payoff l_e,
- from Enter $\Longrightarrow w_e(1-\varepsilon)+\varepsilon\cdot 0$.

Chain store with reputation

One-shot game



- Out is a best response, if $w_e(1-\varepsilon)+\varepsilon\cdot 0\leq l_e$, or

$$\varepsilon > \frac{w_e - l_e}{w_e} =: p^*.$$

- Otherwise, Enter is a best response.

Chain store with reputation

One-shot game

Lemma

In one shot game,

- *If $\varepsilon < p^*$, then the Entrant Enters in each wPBE.*
- *If $\varepsilon > p^*$, then the Entrant stays Out in each wPBE.*

- So, in one-shot case, $T = 1$, small $\varepsilon > 0$ does not change anything

Chain store with reputation
Multiple periods

- It turns out that if T is large, the behavior is dramatically different:
- In each pure strategy wPBE*, the Entrant stays Out except for possibly few periods at the end of the game.

Lemma

- Formal statement.

Lemma

For each $\varepsilon > 0$, there is $K < \infty$ such that for each $T \geq K$, in each pure strategy wPBE, in each $t \leq T - K$,*

- *the Entrant stays Out, and*
- *the Chain Store Fights if Entrant enters.*

Chain store with reputation

Multiple periods

- wPBE* equilibrium (strategies + beliefs)
- After each history, belief $p(h)$ that the Chain store is crazy
  - initially $p(\emptyset) = \varepsilon$,
- $*$: we assume that after each h
  - $p(h, \text{Out}) = p(h)$ (no belief change if Entrant stays Out),
  - $p(h, (\text{Enter, Accomodate})) = 0$
    - we say that Chain store reveals itself to be normal,
    - this can be derived from Bayes formula on path,
  - if $p(h) = 0$, then $p(h, a) = 0$ (once recognized as normal, always normal) and
  - reasonable restrictions (would arise from sequential equilibrium).

Chain store with reputation

Multiple periods: after revealing to be normal

- We seperatately consider three types of histories:
  - $p(h) = 0$,
  - $p(h) > p^*$,
  - $p(h) \in (0, p^*]$.

Chain store with reputation

Multiple periods

- If $p(h) = 0$,
  - Chain store is known (for ever) to be normal,
  - the continuation behavior as in the SPE of complete information
    - Entrant always Enters,
    - Chain store always Accommodates,
    - Chain store gets l_i,
  - next periods beliefs $p(h, a) = 0$ are the same,
- Hence, Chain store continuation payoffs:

$$(T - t + 1)\, l_i.$$

Chain store with reputation

Multiple periods

- If $p(h) > p^*$,
  - Entrant is Out,
  - Chain store payoffs w_i,
  - next periods beliefs $p(h, Out) = p(h) \geq p^*$ are the same,
- Hence, Chain store continuation payoffs

$$(T - t + 1) w_i.$$

- Notice that we use here the fact that each Entrant plays in only one period.
  - otherwise, Entrant could try to test the Chain store to learn whether it is really crazy or normal.

Chain store with reputation

Multiple periods

- Finally, suppose that $p\left(h\right) \in (0, p^{*}]$.
- Remember that we are looking at pure strategy equilibria only (for now).
- Two actions for the *normal* type of the Chain store if Entrant enters:
  - Accommodate,
  - Fight.
- Is it possible to have an equilibrium in which the normal Chain store Accommodates after history h?

Chain store with reputation

Multiple periods

- Suppose yes, and the normal Chain store Accommodates after history h such that $p(h) \in (0, p^*]$.
- Beliefs :
  - $p(h, \text{Enter, Accommodate}) = 0$,
    - only normal type Accommodates,
  - $p(h, \text{Enter, Fight}) = 1!$
    - only crazy type Fights
  - both histories are on-path (so we use Bayes formula)

Chain store with reputation

Multiple periods

- Payoffs and continuation payoffs afrer Entrant enters:
  - if normal Chain store Accomodates
    - today l_i
    - in the next periods, $(T - t)\, l_i$
    - together $(T - t + 1)\, l_i$
  - If normal Chain store deviates and Fights,
    - today 0,
    - in the next periods $(T - t)\, w_i$.
  - If

$$(T - t + 1)\, l_i < (T - t)\, w_i,$$

then Accommodate is not best response!

Chain store with reputation

Multiple periods

- Accommodate is not best response, if

$$(T - t + 1)\, l_i < (T - t)\, w_i,$$

or, after some algebra, if

$$t < T - \frac{l_i}{w_i - l_i}.$$

- Let $K = \frac{l_i}{w_i - l_i}$. In each equilibrium, if $t < T - K$, the Chain store Fights.
- But then, for each $t < T - K$, Entrant stays Out.

Chain store with reputation

Multiple periods

- Value of reputation building.
- Assume that $l_i < \frac{1}{2} w_i$. Then, $K < 1$.
- Complete information payoffs for the Chain store:

$$T l_i,$$

- With reputation:

$$\left(T-1\right) w_i + l_i = \left(T-1\right) w_i + l_i.$$

- Value or reputation is equal to the difference

$$\left(T-1\right)\left(w_i - l_i\right).$$

- When T is large, dramatic effect of the ε-probability craziness for the payoffs.

Chain store with reputation

Multiple periods (mixed strategy equilibria)

- So far, we assumed pure strategies.
- It is possible that there are mixed strategy equilibria. The main result, that Entrant must stay Out for all but a small number of periods does not change.
- We briefly sketch the idea.

Chain store with reputation

Multiple periods (mixed strategy equilibria)

- Suppose that $\sigma(h, E)$ is the probability that the Chain store Fights after history $(h, Enter)$.
- Notice that $\sigma(h, E) \geq p(h)$.
- Moreover, by Bayes formula,

$$p(h, (\text{Enter,Fight})) = \frac{\text{Probability that CS is crazy after} h}{\text{Probability that CS Fights after} h}$$
$$= \frac{p(h)}{\sigma(h)}.$$

- Remember that

$$p(h, \text{Out}) = p(h) = p(h)\frac{1}{\sigma(h, O)},$$

where we denote $\sigma(h, O) = 1$.

Chain store with reputation

Multiple periods (mixed strategy equilibria)

- Hence, for any history where Chain store never Accomodated, we can iterate to obtain

$$
\begin{aligned}
p(h) &= p(h_{t-1}) \frac{1}{\sigma\left(h_{t-1}, a_E^t\right)}. \\
&= \left(p(h_{t-2}) \frac{1}{\sigma\left(h_{t-1}, a_E^{t-1}\right)} \right) \frac{1}{\sigma\left(h_{t-1}, a_E^t\right)} \\
&= \ldots \\
&= p(\emptyset) \prod_{s:a_E^s} \frac{1}{\sigma(h_s, E)}.
\end{aligned}
$$

Chain store with reputation

Multiple periods (mixed strategy equilibria)

- Recall that Entrant Stays Out if $\sigma(h) > p^*$.
- Suppose that there are at least k periods such that Entrant Enters, or such that $\sigma < p^*$. Then

$$p(h) = p(\emptyset) \prod_{s:a_E^s} \frac{1}{\sigma(h_s, E)}$$

$$\geq p(\emptyset) \left(\frac{1}{p^*}\right)^k$$

$$= \varepsilon \frac{1}{p^{*k}}.$$

- But we also have $p(h) \leq 1$. Hence, $1 \geq \varepsilon p^{*-k}$, or, after taking logarithms from both sides, and some algebra (remember that $\log p^* < 0$)

$$k^* \leq \frac{\log \varepsilon}{\log p^*}.$$

Chain store with reputation

Multiple periods (mixed strategy equilibria)

- To summarize, in any (mixed or pure) strategy equilibrium, Entrant must stay Out except for possibly, at most

$$k^* \leq K = \frac{\log \varepsilon}{\log p^*}$$

periods.

Plan

Example

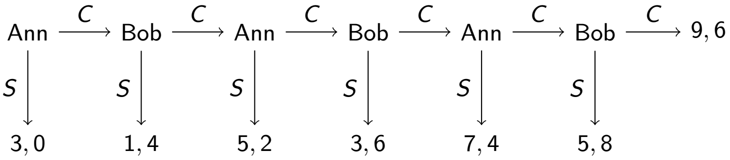
Centipede game:

- Two players, Ann and Bob. There are \$3 on the table.
  - The game lasts for at most $T < \infty$ periods. Each period that the game is still on, the pile on the table increases by \$2.
    - Value of the pile $= 3, 5, 7, ..., 2T + 1$.
  - Players may stop in alternating periods: Ann in odd periods, Bob in even.
  - If game is stopped in period t,
    - players split the pile in half,
    - the stopping player gets \$1.5 from the other player,
    - so, a player who stops in period t gets $t + 2$ and the other player gets $t - 1$.
  - If nobody stops till period T, as if the game stopped n period $T + 1$.

Centipede game

Basic model

- Example: $T = 6$



- Suppose that each player i has one of two types:
  - normal, with probability $1-\varepsilon$: normal payoffs,
  - crazy, with prob. $\varepsilon > 0$, : the crazy type always "Continues".

Lemma

(informal statement) If T is sufficiently large, then there is an wPBE such that both players Continue for a long time, and then Stop randomly.*
The Stopping does not occur before $T - K_\varepsilon$, where K_ε depends on ε, but not on T.

Centipede game

No pure strategy equilibria

- In the next 3 slides, we are going to check that if ε is small, there is no pure strategy equilibrium.
- On the contrary, suppose that player i, say Ann, is the first one to stop and she does it for the first time in period t. We show, in turn that
  - Step 1: Bob prefers to continue till his last period, T or $T-1$
  - Step 2: Ann stops for the first time just before Bob does, i.e., $t \geq T-2$,
  - Step 3: But then Bob prefers to stop in period $t-1$.
  - Contradiction shows that there is no pure strategy equilibrium.

Centipede game

No pure strategy equilibria: Step 1

- Let $t \leq T$ be the first period that a normal player, say Ann, stops.
  - such t must exists as the last normal player will Stop for sure.
- Before t beliefs do not change (because both types of each player play the same action, Continue)..
- But then, after t, Ann, if she Continues, she must be crazy,
- Bob's best response must be to wait till his last period (either T or $T-1$),
  - before t, Bob Continues,, by our assumption on equilibrium
  - after t, ifAnn did not stop, Bob faces crazy type, hence he should continue.

Centipede game

No pure strategy equilibria: Step 2

- So, if Ann were to Continue in period t, Bob waits till period T or $T-1$ (whichever is her action.)
- But then, normal Ann wants to Continue till the last moment before Bob Stops.
- Hence, $t \geq T-2$.

Centipede game

No pure strategy equilibria: Step 3

- Hence, normal Ann stops for the first time in period $t \geq T - 2$.
- Let's check Bob's incentives in period $t - 1$.
  - payoff from Stop in period $t - 1$ is $t - 1 + 2 = t + 1$, and
  - payoff from Continue in period $t - 1$ is at most

$$\varepsilon T + (1 - \varepsilon) t,$$

  - if ε is small, better to Stop.
- We got a contradiction.

Centipede game

Mixed strategy equilibria

- For a mixed strategy equilibrium, we need some notation:
  - t-player: player who moves in period t,
  - p_t - probability that the t-player is crazy,
  - α_t - probability that the normal type of the t-player stops,
  - q_t - probability that the t-player stops:

$$q_t = (1 - p_t)\,\alpha_t.$$

- If $\alpha_t \in (0, 1)$, then the player must be indifferent between stopping and continuing.

Centipede game

Mixed strategy equilibria

- We proceed in the following steps:
- Step 0: There are no gaps:
  - there is K such that $\alpha_t = 0$ for $t < T - K$ and $\alpha_t > 0$ for $t \geq T - K$.
  - $\alpha_t < 1$ unless $t = T - 1, T$.
- Step 1: We show that $\alpha_t > 0$, then it must be that $q_{t+1} = \frac{1}{2}$.
- Step 2: $p_T = \frac{1}{2}$.
- Step 3: $p_t = 2^{-\frac{1}{2}(T-t)}$
- Step 4: $K \simeq 2\left(-\log_2 \varepsilon\right)$.

- The argument in the pure strategy case also implies that if the first period in which each player stops for sure must be their last periods,
- So before that, each player must either Continue for sure, or randomize.
- If $\alpha_t = 0$, i.e., the t-player Continues in period t,
  - then the $(t-1)$-player also wants to Continue (as why to stop if one can stop in 2 periods and get more money),
  - but then $(t-2)$-player wants to Continue, etc.
- Hence, no gaps.

Centipede game

Mixed strategy equilibria: Step 1

- If $\alpha_t \in (0,1)$, then the player must be indifferent between stopping and continuing.
  - payoff from Stopping $t+2$,
  - payoff from Continuing and then Stopping the next time is

$$q_{t+1}\left((t+1)-1\right)+\left(1-q_{t+1}\right)\left((t+2)+2\right).$$

  - If player is indifferent, we have

$$t+2=q_{t+1}\left((t+1)-1\right)+\left(1-q_{t+1}\right)\left((t+2)+2\right),$$

  or

$$q_{t+1}=\frac{1}{2}.$$

- Note that $q_{t+1}<\frac{1}{2}$, then the t-player wants to Continue.

Centipede game

Mixed strategy equilibria: Step 2

- The normal type of T-player must stop for sure in period T, $\alpha_T = 1$.
- Because $\alpha_{T-1} > 0$, we must have $q_T = \frac{1}{2}$
- But,

$$\frac{1}{2} = q_T = 1 - p_T$$

hence,

$$p_T = \frac{1}{2}.$$

Centipede game

Mixed strategy equilibria: Step 3

- Suppose that $\alpha_t > 0$ and that t-player and $T-$player are the same.
- Belief updating in periods t such that $t+1$-player is indifferent: $q_t = \frac{1}{2}$

$$p_{t+2} = \frac{P\left(\text{crazy}, C_t\right)}{P\left(C_t\right)} = \frac{p_t}{1-q_t} = 2p_t,$$

or

$$\begin{aligned} p_t &= \frac{1}{2}p_{t+2} = \frac{1}{4}p_{t+4} = \dots, \\ &= \frac{1}{2^{\frac{1}{2}(T-t)}}. \end{aligned}$$

- Hence, beliefs decrease by half every two periods.
- This cannot continue forever, because $p_t \geq \varepsilon$.

Centipede game

Mixed strategy equilibria: Step 4.

- Let K the highest number period such that $\varepsilon \leq 2^{-\frac{1}{2}K}$.
- We construct equilibrium such that
  - everybody Continues before $T - K$,
  - after $T - K$, each player is randomizing (with an exception below) so that $q_t = \frac{1}{2}$.
  - exception: normal type of T-player stops in period T for sure.
- Notice that
$$K \simeq 2\left(-\log_2 \varepsilon\right).$$
  (remember that $\log \varepsilon < 0$ due to $\varepsilon < 1$).
- In particular, K does not depend on T.

Plan

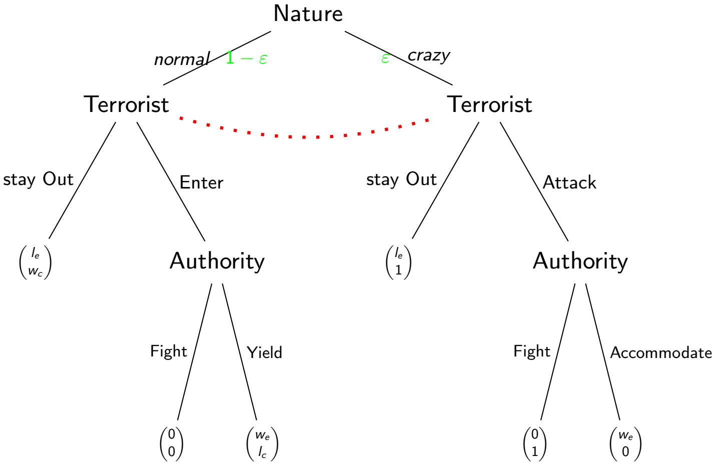
Applications

- “Never negotiate with terrorists.”
- Reputation for promise-keeping.
- Other, or richer reputation types

“Never negotiate with terrorists”

- Two players:
  - authority,
  - terrorist,
- Terroritst decides whether to attack or not,
- Authority decides whether to yield to the demands or not.
- Authority
  - normal, with probability $1 - \varepsilon$,
  - crazy “always Fights” with probability $\varepsilon > 0$.

"Never negotiate with terrorists"



- The same analysis as in the Chain store.

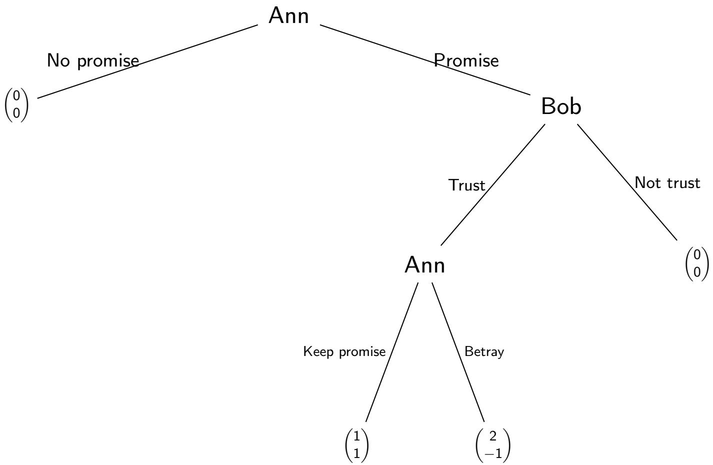
“Never negotiate with terrorists”

Examples of “crazy” strategies: Kissinger and Nixon

Promise-keeping

- Simple model of promise keeping.
- In each period, Ann wants to exchange favors Ann decides whether to or not a promise.
- If Ann makes a promise, Bob chooses whether to trust Ann or not.
- Then, if she is trusted, Ann decides whether to fulfil promise or not.

Promise-keeping



Promise-keeping

- In all SPE
  - always Betray,
  - never Trust,
  - make or not a promise : doesn't matter.

Promise-keeping

- But, if there is a small probability that Ann always Keeps promises:
- $\Rightarrow$ unique reputation equilibrium.

Promise-keeping

Cost of keeping promises

- A small variation of the basic model:
- Prior to decision whether to make a promise, Ann observes the cost of keeping promises $x \sim F(.)$.
  - F is c.d.f. on $\mathbb{R}^+ = \{x : x \geq 0\}$.
- Her payoff from Keeping the promise is $2 - x$
  - prebvious model corresponds to $x = 1$.

Promise-keeping

Cost of keeping promises

- WE can check that, without reputation, unique SPE is that Ann always breaks her promises.
- With reputation, unique SPE:
  - she makes promises only if $x \leq 2$,
  - she always keeps them,
  - and Bob always trusts.

Model of reputation

- What is the right reputation type?
- Sometimes, the right reputation is "bad" because he always Fights.
- Sometimes, the reputation is good" because Continues, or Keeps Promises.
- What if incomplete information is rich?
  - say there are two types, one that always Fights, and
  - the other that always Accommodates,
  - what then?

Model of reputation

- Very easy to extend the basic model.
- The normal type has a choice whom to imitate.
- How does the normal type make her choice?

Model of reputation

- Suppose that payoffs of Ann are $u_A(a, b)$ from Ann's action a and Bob's action b.
- If Ann convinces Bob that she is type a_0, Bob will respond with best response

$$b^*(a) = \arg\max_b u_B(a, b).$$

- Ann's payoff is

$$u_A(a, b^*(a)).$$

- But then, Ann wants to build a reputation for a type that will maximize her payoff

$$\max u_A(a, b^*(a)).$$

- The action that maximizes Ann's payoff is called the Stackelberg action.
  - why? Notice that the same calculation as above is done by the Leader in the Stackelberg version of the Cournot model.

Plan

Conclusions

- Concepts
  - Modelling reputations.
  - wPBE with mixed strategies.
- Skills:
  - Conditions for successful reputation building (payoffs, lentgh of the game, beliefs)
  - Finding reputation equilibria
  - Computing value of reputation