Quality-price monopoly

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The monopolist sells goods (q, p) where q is the quality and p is the price. The profits from selling each unit of such a good are $p - c(q)$, where the $c(q) = \frac{1}{2}q^2$ is the choice of the quality. The monopolist wants to maximize the profits.

The consumer's utility from having a good is equal to

$$\theta q - p.$$

Here, $\theta \geq 0$ is the taste for the quality. The consumer buys the good if the utility from owning it is positive.

Complete information case

We start with perfect information case, where the monopolist knows θ. The profit maximization problem is

$$\begin{aligned} \max_{q,p} p - \frac{1}{2}q^2 \text{ st.} \\ IR : \theta q - p \geq 0. \end{aligned}$$

The *individual rationality* (IR) constraint ensures that the consumer wants to buy the good.

The monopolist will choose as large price as possible. That means that

$$p = \theta q,$$

and the monopolist problem transforms into

$$\max_{q} \theta p - \frac{1}{2}q^2.$$

In particular, we eliminated constraints and (in the same step) reduced the choice variables from 2 (q and p) to 1 (only q).

One easily checks that the optimal quality choice is

$$\begin{aligned} q^{CI}(\theta) = \theta, \\ p^{CI}(\theta) = \theta^2. \end{aligned}$$

Adverse selection

The monopolist faces two types of consumers θ_h (with probability λ) and $\theta_l < \theta_h$ (with probability $1-\lambda$). The monopolist offers a menu of two goods

$$(q_h, p_h) \text{ and } (q_l, p_l).$$

The idea is that the first good will be chosen by the high type and the other by the low type. To ensure that two *incentive compatibility* constraints must be satisfied:

$$\begin{aligned} IC_h &: \theta_h q_h - p_h \geq \theta_h q_l - p_l, \\ IC_l &: \theta_l q_h - p_h \leq \theta_l q_l - p_l. \end{aligned}$$

The IC_h constraint ensures that the high type prefers the high good to the low good. The constraint IC_l ensures that the low type prefers the low good to the high good.

Additionally, both consumers must be better off from trading than from not trading:

$$\begin{aligned} IR_h &: \theta_h q_h - p_h \geq 0, \\ IR_l &: \theta_l q_l - p_l \geq 0. \end{aligned}$$

The monopolist problem is to choose the menu that maximize the profits subject to the individual rationality and incentive compatibility constraints:

$$\begin{aligned} &\max_{q_h, p_h, q_l, p_l} \lambda \left(p_h - \frac{1}{2} q_h^2 \right) + (1-\lambda) \left(p_l - \frac{1}{2} q_l^2 \right) \\ &\text{st. } IC_h, IC_l, IR_h, IR_l \text{ constraints.} \end{aligned}$$

Note that one of the goods that the monopolist can offer is 0 price and quality good $(0,0)$. Such a good effectively corresponds to no-trade. Thus, the menu-choice problem is sufficiently general and allows from excluding some or all types of the buyer from the market.

Incentive compatibility of the first-best menu

We are going to check whether the best menu in the complete information case (sometimes called as the first-best menu) $\left(q^{CI}(\theta), p^{CI}(\theta)\right)$ is incentive compatible. If so, it would be great - such a menu would be the optimal solution to the monopolist problem. The adverse selection would not matter.

First, recall that for each type θ

$$\theta q^{CI}(\theta) - p^{CI}(\theta) = 0,$$

or the utility of each agent is 0. When we compute the utility of the low type who buys the high type good, we obtain

$$\theta_l q^{CI}(\theta_h) - p^{CI}(\theta_h) = \theta_l \theta_h - \theta_h^2 = (\theta_l - \theta_h)\theta_h < 0.$$

Hence, the low type does not want the high type good and the IC_l constraint is satisfied. On the other hand, when we compute the utility of the high type who buys the low type good, we obtain

$$\theta_h q^{CI}(\theta_l) - p^{CI}(\theta_l) = \theta_h \theta_l - \theta_l^2 = (\theta_h - \theta_l)\,\theta_l > 0.$$

The high type gets positive utility from the low type good! Given that the high type utility from the high type good is 0, that means that the IC_h is violated. The complete information menu cannot be a solution in the adverse selection case. The monopolist has to try something else.

As we can see from the above discussion, the problem is that the high type would like to buy the low type good rather than the high type good. There are couple of ways that the monopolist can deal with this problem. One, the monopolist can try to make the high type good more attractive (by, for example, lowering the price). Another one, the monopolist can try to make the low type good less attractive for the high type, by, say, lowering the quality. As we will see, both of these will be necessary to find the optimal solution.

Quality in the incentive compatible menu

We have the following observation.

Lemma. *Suppose that a menu $\{(q_h, p_h), (q_l, p_l)\}$ satisfies the two IC constraints. Then, $q_h \geq q_l$.*

This is a very nice observation that has a simple economic interpretation: higher types (who value quality more) will get higher quality goods.

Proof. After some rearranging the incentive constraints can be rewritten as

$$\begin{aligned} IC_h &: \theta_h (q_h - q_l) \geq p_h - p_l, \\ IC_l &: \theta_l (q_h - q_l) \leq p_h - p_l. \end{aligned}$$

Putting the two inequalities together implies that

$$\theta_h (q_h - q_l) \geq \theta_l (q_h - q_l),$$

or

$$(\theta_h - \theta_l)(q_h - q_l) \geq 0.$$

Because $\theta_h > \theta_l$, it must be that $q_h \geq q_l$ (otherwise, the product of a strictly negative term and a strictly positive term is strictly negative). QED.

Constraints

As in the complete information case, in order to solve the monopolist problem, we want to simplify (or eliminate) the constraints. This is a somehow more complicated process that is best divided into steps. We start with a brief summary of the steps and then explain each step in detail:

1. We show that the IR_h constraint is implied by the other constraints,

2. We show that the IC_h constraint is binding (i.e., it is satisfied with equality).

3. We show that IC_l constraint is implied by the other constraints and the previous observation.

4. Finally, we show that the IR_l constraint is binding.

First, notice that we can eliminate the individual rationality constraint for the high type.

Lemma. *Suppose that the menu satisfies IC_h and IR_l. Then, it also satisfies IR_h.*

Proof. Suppose that the menu satisfies IC_h and IR_l. Then,

$$\theta_h q_h - p_h \geq \theta_h q_l - p_l \geq \theta_l q_l - p_l \geq 0.$$

The first inequality comes from IC_h, the second inequality from the fact that $\theta_h > \theta_l$, and the last from IR_l. Together the inequalities imply that

$$\theta_h q_h - p_h \geq 0.$$

QED.

Next, we are going to show that the IC_h is binding (i.e., it must be satisfied with equality, instead of inequality).

Lemma. *In the monopolist problem, it must be that IC_h constraint is binding, or*

$$p_h = p_l + \theta_h (q_h - q_l).$$

Proof. The IC_h says that

$$p_h \leq p_l + \theta_h (q_h - q_l).$$

If we increase p_h so that the above constraint is satisfied with equality than

- we are going to increase the monopolist's profit,

- we are going to relax the constraint IC_l ("relaxing the constraint" means that it becomes easier to satisfy). This is because this constraint says

$$p_h \geq p_l + \theta_l (q_h - q_l).$$

  If p_h becomes higher, the constraint is still satisfied,

- we are going not to touch the IR_l constraint because it does not depend on p_h.

Because the monopolist likes higher profits, it will increase p_h until the IC_h constraint becomes binding. QED.

Third, we show that IC_l constraint is implied by the other constraints and the previous observations.

Lemma. *If IC_h constraint is binding, and $q_h > q_l$, then IC_l constraint holds.*

Proof. Notice that

$$\theta_l\left(q_h - q_l\right) \leq \theta_h\left(q_h - q_l\right) = p_h - p_l.$$

The first inequality comes from the fact that $q_h > q_l$ and $\theta_l < \theta_h$. The equality comes from the fact that IC_h constraint is binding. Together, the the inequality and equality imply the IC_l constraint. QED.

Finally, we show that the IR_l constraint is binding.

Lemma. *In the monopolist problem, it must be that IR_l constraint is binding, or*

$$p_l = \theta_l q_l.$$

Proof. If not then we can always replace p_l by $p_l + \varepsilon$ and p_h by $p_h + \varepsilon$, where $\varepsilon \leq \theta_l q_l - p_l$. Doing so (a) is not going to affect the constraint IC_h (because it does not change the difference between the two prices $p_h - p_l$), and (b) is going to increase the monopolist's profits. Hence, good idea. QED.

Simplified monopolist's problem

The above discussion allows us to simplify the monopolist problem. The binding constraints imply that

$$\begin{aligned}
&p_l = \theta_l q_l, \text{ and} \\
&p_h = p_l + \theta_h\left(q_h - q_l\right) = \theta_l q_l + \theta_h\left(q_h - q_l\right) = \theta_h q_h - \left(\theta_h - \theta_l\right) q_l.
\end{aligned}$$

We can substitute this equalities into the monopolist profit function. In this way, we eliminate prices and the two constraints. Thus, we obtain

$$\begin{aligned}
&\max_{q_l, q_h} \lambda\left(p_h - \frac{1}{2}q_h^2\right) + (1-\lambda)\left(p_l - \frac{1}{2}q_l^2\right) \\
=&\max_{q_l, q_h} \lambda\left(\theta_h q_h - \left(\theta_h - \theta_l\right) q_l - \frac{1}{2}q_h^2\right) + (1-\lambda)\left(\theta_l q_l - \frac{1}{2}q_l^2\right).
\end{aligned}$$

Note: If you read carefully the discussion of the constraints above, you will notice that there is one constraint that we did not explicitly use in the statement of the monopolist's problem. Namely, the IC constraints imply that

$$q_h > q_l.$$

However, if you look at the solution below to the monopolist problem, you can see that the solution satisfies the above condition. Hence, this constraint is satisfied at the optimum, and we do not have to correct for it.

We proceed to solve the above problem. The first order conditions with respect to $q_h$imply that

$$\lambda\left(\theta_h - q_h\right) = 0,$$

which implies

$$q_h^* = \theta_h.$$

In particular, the optimal quality for the high type good is the same as in the first-best case. Further, the first-order conditions with respect to q_l say that

$$-\lambda\left(\theta_h - \theta_l\right) + \left(1 - \lambda\right)\left(\theta_l - q_l\right) = 0,$$

or

$$q_l^* = \theta_l - \frac{\lambda}{1 - \lambda}\left(\theta_h - \theta_l\right) \leq \theta_l = q^{CI}\left(\theta_l\right).$$

The optimal quality of the low type good is lower than in the complete information case.

What happens to the utilities? Because the IR_l condition is binding, the low type gets 0 utility, the same as in the complete information case. On the other hand, the high type's utility is equal to

$$\theta_h q_h^* - p_h = \theta_h^2 - \left(\theta_h \theta_h - \left(\theta_h - \theta_l\right) q_l^*\right) = \left(\theta_h - \theta_l\right) q_l^* > 0,$$

which is strictly positive. The high type is happy!

Summary

To summarize. The first-best menu is not incentive compatible, hence it cannot be used by the monopolist under adverse selection. The problem is that the high type would prefer to choose the low type good instead of the high type good designed by the monopolist. To solve the problem, the monopolist chooses a different menu, in which the high type pays a lower price, and the low type receives a lower quality good. The low type gets the same utility as in the complete information case (which is 0). The high type receives a strictly higher utility.