1. Suppose that the true state of the world is a temperature $\omega \in Z$ (expressed in integer degrees of Celsius). I have a phone app that tells me whether the temperature is strictly larger than 0 or weakly smaller than 0. I have no other information. Thus, my information structure is given by partition $\{\{..., -2, -1, 0\}, \{1, 2, 3, ...\}\}$, or by the types

$$T_M(\omega) = \begin{cases} \{..., -2, -1, 0\}, & \text{if } \omega \leq 0, \\ \{1, 2, 3, ...\}, & \text{if } \omega > 0. \end{cases}$$

(Here, M stands for Marcin)

(a) Show that $K_M\{5\} = \emptyset$. In other words, in no state of the world, "I know that the temperature is 5."

(b) Suppose that $E = \{\omega : \omega \geq 3\}$. E is the event that corresponds to "The temperature is larger or equal to 3." Show that in no state of the world, I know that event E is true.

(c) Suppose that $F = \{\omega : \omega < 5\}$, i.e., F means "the temperature is below than 5. Explain that

$$K_M F = \{..., -2, -1, 0\}.$$

Or, I know that the temperature is below 5, whenever the true temperature is 0 or below.

Solutions: Part(a) Observe that $K_M\{5\}$ is the set of states ω such that $T_M(\omega) \subseteq \{5\}$. But my type of information is never contained in $\{5\}$ (my infromation is to imprecise). Hence, the set of states where I know that the tempereture is 5 is empty.

Part (b)

2. Convince yourself that for each information structure,

$$K_i T(\omega) = T(\omega).$$

In other words, each agent knows their type. Also, convince your self that

$$K_i \Omega = \Omega.$$

In other words, each player always knows that the world exists.

Solutions: Recall that for each event E,

$$K_i(E) = \{\omega : T_i(\omega) \subseteq E\}.$$

Fix state ω_0. Then,

$$K_i(T(\omega_0)) = \{\omega : T_i(\omega) \subseteq T_i(\omega_0)\} = T_i(\omega).$$

The first equality is a definition of the knowledge event; the second equality comes from the fact that the information structure is a parttion, and the only way one type can be included in the other one is if the two types are equal.

3. We can compare different information structures in many different ways. For example, there is a natural notion of "better information". Suppose that there are two people, Alice and Bob. Recall that information structure is a partition of the state space. We say that Bob has *better information* than Alice, if Bob's partition is *finer* than Alice's. The latter notion should be intuitive: any element of Bob's partition is contained in some element of Alice's partition: For eacg $\omega \in \Omega$, $T_B(\omega) \subseteq T_A(\omega)$.

   (a) Explain that for any event E,

   $$K_A E \subseteq K_B E.$$

   In other words, if Alice knows E, then Bob knows E as well.

   Solutions: Notice that for each ω, if $\omega \in K_A E$, then $T_A(\omega) \subseteq E$ (by definition of the knowledge event). Because $T_B(\omega) \subseteq T_A(\omega)$, it must be also that $T_B(\omega) \subseteq E$. Hence, $\omega \in K_B E$.

   (b) The converse does not hold necessarily. In other words, give an example of information structures of Alice and Bob, where Bob has a better information, and an event E such that

   $$K_B E \nsubseteq K_A E.$$

   Solutions: Suppose that $\Omega = \{1, 2\}$. Alice does not know anything and Bob knows the state precisely:

   $$\mathcal{T}_A = \{\Omega\},$$
   $$\mathcal{T}_B = \{\{1\}, \{2\}\}.$$

   . Let $E = \{1\}$. Then, $K_B E = \{1\}$ and $K_A E = \emptyset$.

4. There are four roads leading to the capital city from four directions of the world: West, South, East and North. There are four invading armies approaching the city along each of the roads. The commander of each army sees the state of the gate in front of him as well as the gate immediately to the left and to the right of him. Each gate can be either weakly or strongly defended.

   (a) Describe the state space. Describe the information structure of general West.

Solutions: There are 16 states of the world and $\Omega = \{0000, 0001, 0010, 0011, ..., 1110, 1111\}$, with the interpretation that $x_1x_2x_3x_4$ corresponds to the state where the gatein front of general i is weak if $x_i = 0$ and strong if $x_i = 1$.

(b) After arriving in front of the city, each commander raises a red flag if at least two of the gates she observes are strongly defended. Otherwise, she raises a green flag. The color of the flag is observed by the neighboring armies.
Suppose that the general West observes general North raising the red flag and general South raising the green flag. Describe how this additional information changes the state space and the information structure of the commander West.

Solutions: The North red flag means that there are at least two ones among Western, Eastern, and Northern gates. The South green flag means that there is at most one 1 among Western, Southern, and Eastern gates.

The only states that are consistent wiht it are

- 1001,
- 0011.

Because general West can distuinguish these two states, her information structure is $T' = \{\{1001\}, \{0011\}\}$.