1. (O.J. Simpson trial) The O. J. Simpson murder case was a famous criminal trial in which former football star O.J. Simpson was tried for murder of his wave and a on two counts of murder for the June 12, 1994, deaths of his ex-wife Nicole Brown Simpson and Mezzaluna restaurant waiter Ronald Goldman. During the trial, the prosecution brought up the fact that O.J. investigated by police for domestic violence multiple times and pleaded no contest to spousal abuse in 1989. The defense argued that past domsetic abuse is irrelevant because only 0.1% of the men who physically abuse their wives actually end up murdering them.
   It turns out that the defense argument is not valid, but to explain why, we need to make computations based on the Bayes formula. Let G be the event that O.J. Simpson was guilty of the murder, let A be the event that O.J. Simpson abused his wife. We assume that

$$
\begin{aligned}
&(a) P(M|G) = P(M|G,A) = 1, \\
&(b) P(M|G^c,A) = P(M|G^c) = \frac{1}{30000}, \\
&(c) P(G|A) = \frac{1}{1000}, \\
&(d) P(G^c|A) = \frac{999}{1000}.
\end{aligned}
$$

   Here (a) is just logic, (b) the second equality comes from the following data: in 1994, around 5000 women were murdered, 1500 by their husbands, and given population of 100 mln wifes, we get the probability that a wife is killed given her husband being innocent as $\frac{5000-1500}{100mln} \approx \frac{1}{30000}$; the first equality comes from the fact that information that husband was abusive should not affect the murder rates if the husband was not the one who did it, (c) and (d) come from the defense argument. The defence was explaing that A is irrelevant because $P(G|A)$ is very small. However, that is not relevant number, because additionally to the fact that O.J. abused his wife, we also know that she was murdered. Hence, the relevant number is

$$P(G|A,M).$$

   Let us calculate.

   (a) Assume that the probability that a wife is abused by her husband is μ (there are statistics about it, but we do not need to know μ to make our argument). Use the Bayses formula and teh above data to compute

$$P(G,A).$$

   (The answer should depend on μ.)

   Solutions:

$$P(G,A) = P(G|A)\,P(A) = \frac{1}{1000}\mu.$$

(b) Use the Bayes formula and the answer to question (b) to compute

$$P(M, G, A).$$

(Once more, the answer should depend on μ.)

Solutions:

$$P(M, G, A) = P(M|G, A)\, P(G|A)\, P(A) = \mu \frac{1}{1000},$$

(c) Use Bayes formula and the above data to compute

$$P(A, M) = P(M, G, A) + P(M, G^c, A).$$

(The answer should depend on μ.)

Solutions: Notice that

$$P(M, G^c, A) = P(M|G^c, A)\, P(G^c|A)\, P(A) = \mu \frac{1}{30000} \frac{999}{1000}.$$

Thus,

$$P(A, M) \approx \mu \left(\frac{100}{100000} + \frac{3.33}{100000} \right) = \frac{103.33}{100000} \mu.$$

(d) Finally, use the above answers to compute

$$P(G|A, M).$$

Does the answer depend on μ? Does the answer support or refute the defense argument.

Solutions:

$$P(G|A, M) = \frac{P(G, A, M)}{P(A, M)} = \frac{\mu \frac{1}{1000}}{\frac{103.33}{100000} \mu} = \frac{100}{103.33} \approx 97\%.$$

That is not a small number.

2. (Investigation) Consider the following version of the Investigation. Trump is guility for sure, but there is uncertainty whether Mueller has evdience and whether Trump knows about it. There are three states of the world:

- n: no evidence,
- eu: there is evidence, but it is unknown to Trump,
- ek: there is evidence and it is known to Trump.

The information structure is

$$\mathcal{T}_{Mueller} = \{\{n\}, \{eu, ek\}\},$$
$$\mathcal{T}_{Trump} = \{\{n, eu\}, \{ek\}\}.$$

(a) Describe the information structure on a well-labeled picture.

(b) Suppose that both players have a prior belief that there is evidence with probability p and, if there is evidence, Trump will know about it with probability q. The following table describes the beliefs of each type

Use

Trump	n	eu	ek
U	$\frac{1-p}{1-p+(1-q)p}$	$\frac{(1-q)p}{1-p+(1-q)p}$	0
K	0	0	1

,

Mueller	n	eu	ek
N	1	0	0
E	0	$1-q$	q

.

Bayes formula to explain Trump's beliefs.

Solutions:

$$P(n|U) = \frac{P(n, U)}{P(U)} = \frac{P(n)}{P(n) + P(eu)} = \frac{1-p}{1-p+p(1-q)}.$$

$$P(eu|U) = 1 - P(n|U).$$

(c) Mueller needs to decide whether to rush and finish the investigation or continue. The latter means that he may get extra evidence, but he also risks that Trump closes off the investigation before it is finished. Trump needs to decide whether to wait or to start the prodcedure that leads to Mueller dismissal. The payoffs depend on whether Mueller has the evidence and they are in the table below:

$\omega = n$:Mueller\Trump	Wait	Fire Mueller
Finish	$1, 1$	$1, 0$
Continue	$0, 2$	$0, -5$

,

$\omega = eu, ek$:Mueller\Trump	Wait	Fire Mu
Finish	$2, -5$	$0, 0$
Continue	$3, -10$	$1, -$

Explain that Mueller has strictly dominant action for each of his types.

Solutions: Mueller knows which of the payoff matrices is the true one. If $\omega = n$, Finish is strictly dominant (with payoffs 1 vs 0 and 1 vs 0). If there is evidence, Continue is a strictly dominant action (with payoffs 3 vs 2 and 1 vs 0).

(d) Given Mueller's behavior, what is Trump's best response? How does your answer depend on p and q?

Solutions: In state ek, Trump knows that the right payoff matrix is being played. In this case, Firing Mueller is strictly dominant (0 vs -5 and -5 vs -10).

If Trum has type U, then, given Mueller strictly dominant strategy, Trump's expected payoff from waiting is

$$\begin{aligned} & P(\omega = n|U)\, u_T(F, W|\omega = n) + P(\omega = eu|U)\, u_T(C, W|\omega = eu)\,. \\ = & \frac{1-p}{1-p+(1-q)\,p} 1 + \frac{(1-q)\,p}{1-p+(1-q)\,p}(-10)\,. \end{aligned}$$

The expected payoff from Firing Mueller is

$$\begin{aligned} & P(\omega = n|U)\, u_T(F, FM|\omega = n) + P(\omega = eu|U)\, u_T(C, FM|\omega = eu)\,. \\ = & \frac{1-p}{1-p+(1-q)\,p} 0 + \frac{(1-q)\,p}{1-p+(1-q)\,p}(-5)\,. \end{aligned}$$

Thus, waiting is the best response if

$$\frac{1-p}{1-p+(1-q)\,p} - 5\frac{(1-q)\,p}{1-p+(1-q)\,p} = 6\frac{1-p}{1-p+(1-q)\,p} - 5 \geq 0.$$