

1. (Cheap talk between the CEO and the Investor). The CEO observes the quality of the project $\omega = \{0, 1\}$, and uses messages $m = \{0, 1\}$ to communicate the information to the Investor. The Investor chooses how much to invest $a \in [0, 1]$ into the project. The payoffs of the CEO and the Investor are

$$U_{CEO}(a, \omega) = -(a - (\omega + t))^2,$$

$$U_{Investor}(a, \omega) = -(a - \omega)^2.$$

- (a) Suppose that the Investor belief is given by $p(0), p(1) \geq 0$ such that $p(0) + p(1) = 1$. Find the best response action.

Solutions: The expected payoff is $E_p U(a, \cdot) = -\sum_{\omega} (a - \omega)^2 p(\omega)$. Investor chooses a to maximize the expected payoff

$$\max_a E_p U(a, \cdot).$$

FOC:

$$\begin{aligned} 0 &= \frac{d}{da} \left(-\sum_{\omega} (a - \omega)^2 p(\omega) \right) \\ &= -\sum_{\omega} \left(\frac{d}{da} (a - \omega)^2 \right) p(\omega) \\ &= -\sum_{\omega} 2(a - \omega) p(\omega) \\ &= -2a \sum_{\omega} p(\omega) + 2 \sum_{\omega} \omega p(\omega) \\ &= 2(E_p \omega - a). \end{aligned}$$

Hence, $a_{opt} = E_p \omega = \sum_{\omega} \omega p(\omega)$.

- (b) For what values of t the truthful communication strategy is an equilibrium strategy?

Solutions: It must be that the 0 type of the CEO does not want to deviate and send message 1.

IN truthful equilibrium, $a(\omega) = \omega$. Hence, we need

$$U_{CEO}(a(0), 0) \geq U_{CEO}(a(1), 0).$$

But $U_{CEO}(a(0), 0) = -(0 - (0 + t))^2 \geq U(a(1), 0) = -(1 - (0 + t))^2$. Hence,

$$t^2 \leq (1 - t)^2,$$

or $t \leq \frac{1}{2}$.

2. Ann discovers that one of her two employees, Bob or Celine, is leaking trade secrets to the competition. She estimates the probability that Bob thinks that the probability that Bob is guilty is $\pi \in (0, 1)$. She knows that another employee, Damian, knows who is leaking. She wants to know whether she can trust Damian's report. The payoffs are in the table:

Ann \ Damian	$\omega = \text{Bob}$	$\omega = \text{Celine}$
Fire Bob	$1, t - v_B$	$0, -v_B$
Fire Celine	$0, -v_C$	$1, t - v_C$

Here, v_B, v_C are measures of how much Damian likes, respectively, Bob and Celine, and $t > 0$ is the Damian's satisfaction from the right thing being done.

- (a) What is Ann's best response without any communication with Damian?

Solutions: It is to fire Bob if $\pi \geq \frac{1}{2}$. If $\pi \leq \frac{1}{2}$, Fire Celine.

- (b) Show that if $|v_B - v_C|$ is small enough, then truthful communication is possible in the equilibrium (wPBE). Is it unique equilibrium?

Solutions: If Damian communicates truthfully, Ann's best response is to follow his recommendation. If Ann follows his recommendation, Damian's best response is to communicate truthfully if

$$\begin{aligned} t - v_B &\geq -v_C, \\ t - v_C &\geq -v_B. \end{aligned}$$

(The first inequality ensures that Damian wants to tell the truth if Bob is the guilty one, and the second inequality ensures truthtelling is Celine is guilty.) It follows that

$$-t \leq v_B - v_C \leq t.$$

No, it is not the only equilibrium. There are always babbling equilibria. For instance, when Damian sends a randomly chosen message with the distribution that does not depend on the state.

- (c) Find an equilibrium (wPBE) if $v_B - v_C > t$. Be careful about off-path beliefs.

Solutions: Any babbling equilibrium. If messages are chosen randomly, all messages are on-path.

In pure strategy babbling equilibria, we can assume that the beliefs after off-path message are the same as the beliefs after on-path message (hence, the same as the prior).