1. (Spence's education) Consider the Spence's model of education with the cost function $c(e, \theta) = e(1 - \theta)$, where the ability type $\theta \in \{\theta_h, \theta_l\}$and $0 < \theta_l < \theta_h < 1$.

   (a) Check that the marginal cost of education is decreasing with the ability.

   Solutions:

   (b) Describe the set of all pooling equilibria.

   Solutions:

   (c) Describe all separating equilibria.

   Solutions:

2. (Gazelle stotting) A gazelle notices a tiger creeping in the bush. The tiger wonders whether the gazelle is fast (type f, with probability π) or slow (type s). The gazelle decides whether to start stotting or not. The tiger observes the gazelle's behavior and chooses whether to chase the gazelle or not. The payoffs of the tiger depend on the type of the gazelle and they are equal to

Tiger's payoffs	f	s
chase	-1	2
no chase	0	0

   (It is easier to catch the slow gazelle. The fast gazelle has a sufficiently high chance of running away, which ends with a waste of energy for the tiger.) The gazelle pays cost $a > 0$ if it is chased. Additionally, if the gazelle stots, it pays t_θ (in the energy expenditure). We assume that $t_f < t_s < a$, or that the fast gazelle pays less in the cost of stotting.

   (a) Does the game have pooling equilibria? Is there a pooling equilibrium, where both types of the gazelle are stotting?

   Solutions: Suppose that $\pi(-1) + (1 - \pi) 2 \geq 0$. Then, it is worthwhile to chase the "average" gazelle. Suppse that none of the types stots. This can be an equilibrium, provided that the off-paths beliefs after stotting are that the gazelle is "average" or slow.
   If $\pi(-1) + (1 - \pi) 2 \leq 0$, then, there is a pooling equilibrium, where no gazelle stots, and tiger does not chase.
   If $\pi(-1) + (1 - \pi) 2 \leq 0$, then there is a pooling equilibrium, where both types of gazelle stot, the tiger does not chase and if somebody does not stot, it is believed to be slow and chased.

(b) Does the game have fully separating equilibria?

Solutions: Suppose that the fast and slow gazelle are choosing different behavior. The tiger is not going to chase if she believes that the gazelle is fast., But then, the slow gazelle will try to mimick the fast one. Not an equilibrium.

(c) Suppose that $\pi(-1) + (1-\pi)2 > 0$. Does the game have a partially separating equilibria? Carefully describe the strategies and beliefs.

Solutions: Yes, there is an equilibrium, where the fast gazelle always stots, the slow stots with probability $\alpha \in (0,1)$, the tiger always chases the non-stotting gazelle, and the tiger chases the stotting gazelle with probability $\beta \in (0,1)$.
If the tiger observes the gazelle stotting, it assigns believes

$$q = \frac{\pi}{\pi + (1-\pi)\alpha}.$$

that the gazelle is fast.
The tiger is indifferent between chasing and not chasing the stotting gazelle if

$$q(-1) + (1-q)2 = 0.$$

The tiger prefers to chase the stotting gazelle. Thus, $q = \frac{1}{3} = \frac{\pi}{\pi+(1-\pi)\alpha}$, and $\frac{1-\pi}{\pi}\alpha = 2$, or $\alpha = 2\frac{\pi}{1-\pi}$. (Notice that $2\frac{\pi}{1-\pi} < 1$ due to $\pi(-1) + (1-\pi)2 > 0$.)
The slow gazelle is indifferent between the stotting or not if

$$-a = \beta(-a) - t_s.$$

This implies

$$1 - \frac{1}{a}t_s = \beta.$$

The fast gazelle prefers to stot because the payoff from stotting is equal to

$$\beta(-a) - t_f = \left(1 - \frac{1}{a}t_s\right)(-a) - t_f = -a + t_s - t_f > -a,$$

where the latter is the payoff from non-stotting.

3. (Lark's singing). A skylark notices a falcon hovering in the air. The falcon wonders whether the skylark is healthy (type h, with probability $\frac{2}{3}$) or sick (type s). The skylark decides whether to run away immediately or sing first. The falcon observes the skylark's behavior and chooses whether to attack or not. If the falcon does not attack, it receives payoff 0. If it attacks, it receives payoff 3 if the skylark is sick and payoff -1 if the skylark

is healthy. The skylark's payoffs are described in the table.

Payoffs	not attacked	attacked, type h	attacked, type s
singing	1	$-d_f$	$-d_s$
rnning away	1	0	0

where $d_f < d_s$ is the decrease in the survival chance caused by not running away immediately.

(a) Does the game have pooling equilibria? Carefully describe all off-path beliefs.

Solutions: If nobody sings, the falcon attacks. It is an equilibrium with appropriately chosen off-path beliefs. There is no pooling equilibrium with both types singing.

(b) Does the game have fully separating equilibria?

Solutions: Suppose that both types of skylark are choosing different behavior. The falcon is not going to attack the skylark with the behavior chosen by the healthy type. But then, the sick skylark will try to mimic the healthy one. Not an equilibrium.

(c) Show that there exists an equilibrium, in which the healthy skylark always sings and the falcon always attacks if the skylark runs away immediately. Carefully describe the strategies and beliefs.

Solutions: Yes. In such an equilibrium, the healthy skylark always sings, and the sick one sings with probability $\alpha \in (0, 1)$. The falcon always attacks the non-singing skylark, and attacks the singer with probability β. The falcon's beliefs that the singing skylark is healthy are

$$p = \frac{\frac{2}{3}}{\frac{2}{3} + \frac{1}{3}\alpha} = \frac{2}{2+\alpha}.$$

The falcon is indifferent between attacking the singing skylark if

$$p(-1) + (1-p)\,3 = 0.$$

This implies that $p = \frac{2}{2+\alpha} = \frac{3}{4}$, which implies $\alpha = \frac{2}{3}$. The falcon always attacks the non-sgning skylark, because such skylark is clearly sick.

The sick skylark is indifferent between singing and running away if

$$0 = \beta(-d_s) + (1-\beta),$$

or $\beta = \frac{1}{1+d_s}$. The healthy skylark wants to sing because the payoff from singing is equal to

$$\beta(-d_h) + (1-\beta) = 1 - \beta(1+d_f) = 1 - \frac{1+d_f}{1+d_s} > 0,$$

where the latter is a payoff from non-singing.