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# Welfare Analysis of Dynamic Pricing

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**Abstract.** Dynamic pricing is designed to increase the revenues or profits of firms by adjusting prices in response to changes in the marginal value of capacity. We examine the impact of dynamic pricing on social welfare and consumers' surplus. We present a dynamic pricing formulation designed to maximize welfare and show that the welfare-maximizing dynamic pricing policy has the same structural properties as the revenue-maximizing policy. For systems with scarce capacity, we show that the revenue-maximizing dynamic pricing policy and the market-clearing price are both asymptotically optimal for welfare. We also find in most cases that revenue-maximizing dynamic pricing improves consumers' surplus compared to the revenue-maximizing static price. Our findings can potentially transform the public image of dynamic pricing and provide new managerial insights as well as policy implications: (1) in large-scale systems with scarce capacity, a central planner would essentially implement the same pricing policy as a firm with monopoly power; (2) the revenue-maximizing dynamic pricing policy can benefit customers when the demand elasticity is in a small bounded interval as is the case for several important demand functions.

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## 1. Introduction

Dynamic pricing and revenue management practices have proliferated since the deregulation of the United States airline industry in the late 1970s. Airlines designed a variety of fares and restrictions to appeal to broad market segments and dynamically managed the set of offered fares to maximize expected revenues from fixed capacities. These practices later expanded to hotels, car rentals, and the entertainment industry to the point where most firms in the travel and leisure industries arguably cannot survive without using dynamic pricing. Dynamic price is also used in fashion retailing, especially when capacity cannot be adjusted within the sales horizon.

As dynamic pricing penetrates new industries, executives are raising the question of how dynamic pricing affects consumers and social welfare (firm's profits plus consumers' surplus). They are concerned that dynamic pricing may help firms in the short run but hurt them in the long run if consumers find less value in the companies' products. Indeed, the public and the media appear to believe that firms earn more revenues using dynamic pricing only by extracting from consumers' surplus. As an example, after Disney introduced a dynamic pricing scheme for theme

parks in 2016, Bloomberg published an article, titled "Surge Pricing at Disneyland Could Boost Ticket Costs by 20%" (Palmeri 2016), indicating that customers may have to pay more on average under the new policy. This "zero-sum" hypothesis, while disproved in other contexts such as third-degree price discrimination (Varian 1985, Cowan 2012), has not been fully studied in the context of dynamic pricing. Firms may therefore be reluctant to implement dynamic pricing for fear of consumer backlash and negative press coverage.

In addition to managers' concerns about the public perception of dynamic pricing, regulators may also be interested in the question of whether dynamic pricing reduces social welfare. If it is the case, then the legitimacy of the practice may be questioned by advocacy groups. Regulators may also be interested in developing dynamic pricing policies with the explicit goal of maximizing social welfare instead of profits or revenues. This problem is relevant to the central allocation of scarce resources and to the determination of tolls for infrastructure projects such as bridges and tunnels. They may also wish to compare policies designed to maximize social welfare to those designed to maximize profits, to make a decision to deter (resp., encourage) private infrastructure if the two policies are significantly different (resp., sufficiently similar).

### 1.1. Contributions and Main Results

We consider a central planner's problem of maximizing the total social welfare that can be obtained from  $c$  units of capacity over a finite sales horizon  $[0, T]$ . The firm's problem of maximizing revenues has been studied by Gallego and van Ryzin (1994); the difference of our study is that central planners are interested in maximizing the sum of consumers' surplus and firm's revenues over the sales horizon. We show that the welfare-maximizing dynamic pricing policy shares the same structural properties as the revenue-maximizing pricing policy derived by Gallego and van Ryzin (1994). More precisely, both pricing policies are increasing in the time-to-go and decreasing in the current inventory level. Moreover, when capacity is sufficiently scarce, the market-clearing price asymptotically maximizes both revenue and social welfare as demand and capacity are scaled in proportion. Under certain conditions, even consumers' surplus is asymptotically maximized by the market-clearing price. Thus, in scaled systems, the objectives of the firm, the customers, and the central planner are essentially aligned when capacity is relatively scarce.

In industries without central planners, it is unlikely that firms will price their products to maximize social welfare. However, they may also be reluctant to use the revenue-maximizing dynamic pricing policy and instead use the revenue-maximizing static price, either because managers believe in the zero-sum hypothesis themselves or out of fear that their consumers subscribe to this belief. While it is clear that dynamic pricing benefits firms compared to the static price, does it in fact reduce consumers' surplus as some fear? Our analysis indicates the contrary: in many cases, compared to an optimal static price, dynamic pricing is win-win for many demand functions. More precisely, it increases the expected revenue for the firm and benefits customers by increasing their expected total surplus. This is because the social welfare allocated to the firm and the consumer, conditioning on a sale, is often well aligned. We provide indications of how the elasticity of demand relates to the alignment of these objectives.

We make three main contributions. First, we conduct a welfare analysis on the classic dynamic pricing model developed in Gallego and van Ryzin (1994). We find pricing policies that are asymptotically optimal for revenue, social welfare, and consumers' surplus, respectively. When capacity is scarce, we show that the market-clearing price performs well for both the firm and the central planner, and in some important cases, for the customers as well.

Second, we show that contrary to common belief, the revenue-maximizing dynamic pricing policy improves both the revenue and consumers' surplus over the revenue-maximizing static pricing policy for many

common demand functions. The elasticity of the demand plays an important role in the impact of dynamic pricing on customers.

Third, our results may change the public image of dynamic pricing and have policy implications. We show that the dynamic pricing policy that maximizes revenues is asymptotically optimal for welfare when capacity is scarce. Hence, the use of dynamic pricing should be accepted, if not welcomed, by regulators and the public. This finding also has implications for policy makers regulating private infrastructure as they can be assured that if capacity is sufficiently constrained, then the objectives of the firm are well aligned with those of policy makers.

Section 2 conducts a welfare analysis on dynamic pricing. We review the basic concepts of consumers' surplus, and formulate the optimality equation for the welfare function. We also study fluid approximation, and show the relationship between the welfare-maximizing and revenue-maximizing pricing policies. In Section 3, we define the Pareto efficient frontier of the welfare allocation between the firm and customers over the sales horizon. In Section 4, we show that dynamic pricing is win-win for the constant elasticity demand function analytically and for the exponential demand function numerically. For the linear demand function, dynamic pricing is win-win except in extreme situations where there is sustained demand over an extremely large booking horizon and capacity is very low. We then provide conditions on the demand function under which dynamic pricing is likely to be win-win. Section 5 extends the classic dynamic pricing to accommodate network revenue management and queueing problems such as toll roads and bridges.

### 1.2. Literature Review

The first stream of literature relates to our paper through the concepts of consumers' surplus and welfare. Consumer's surplus is the benefit that a customer gains from participating in market transactions. When aggregated over all of the customers in the market, consumers' surplus equals to the area to the right of the price under the demand curve (see, e.g., Varian 2010). The concept originates in Dupuit (1844) and is essential to consumer theory and welfare economics. In the operations management and marketing literature, the consideration of consumers' surplus is usually made from the perspective of a single customer. The consumer decides whether or not to buy a single unit of the good. In this case, consumer's surplus is usually defined for a single customer as the difference between her willingness to pay (WtP) and the price charged by the firm. For example, in Dhebar and Oren (1985), the customer only purchases the product if her WtP is greater than or equal to the price. The same concept is used by later papers such as Cachon and

Swinney (2009), Su (2007), and Levin et al. (2009) to derive customers' optimal strategy. We refer the readers to Wilson (1993) and references therein for earlier examples. In this paper, we use both interpretations of consumers' surplus, which turn out to be consistent in our setting.

There is a vast literature on dynamic pricing in the revenue management community. Gallego and van Ryzin (1994) investigate the optimal pricing policy of a firm selling a given stock of items over a finite horizon. Since then, dynamic pricing has been analyzed in various market environments including inventory considerations and competition. See Elmaghraby and Keskinocak (2003) and Bitran and Caldentey (2003), and references therein, for (dynamic) pricing models in revenue management. Another stream of literature, such as Su (2007), Aviv and Pazgal (2008), Su (2007), Liu and van Ryzin (2008), Gallego et al. (2008), Chen and Farias (2018), and Chen et al. (2018), and references therein, considers strategic customers. These papers assume that consumers are usually surplus maximizers and focus on finding the optimal pricing policy rather than analyzing the welfare gain from dynamic pricing.

The study of the impact of pricing policies on consumers has a long history. The welfare analysis of third-degree price discrimination in the case of market segmentation has been documented by, e.g., Varian (1985), Cowan (2007), and Cowan (2012). In general, price discrimination hurts social welfare unless the total output increases after price discrimination (Varian 1985). However, there are exceptions and price discrimination can be win–win (Cowan 2012). The pricing policy we are investigating applies to a different setting from market segmentation, and we find that welfare is more likely to increase after the firm switches from a static pricing policy to dynamic pricing. In terms of the pricing of electricity, Borenstein and Holland (2003) and Borenstein (2005) show that the competitive electricity market is inefficient unless the retail price can vary in time as frequently as the production/wholesale costs, which is referred to as real-time pricing. As real-time pricing shares a similar form to dynamic pricing, their papers are in line with our conclusion that dynamic pricing improves social welfare in general. For brick-and-mortar retail, Stamatopoulos et al. (2017) show that dynamic pricing is win–win in the EOQ setting because it reduces holding costs and passes them through to benefit consumers. In the case of platforms setting prices and wages to match demand and supply, Cachon et al. (2017) show that dynamic pricing and dynamic wages in response to different demand states can maximize the platform's profit and increase consumers' surplus compared to a fixed contract. Feldman et al. (2016) conduct an empirical study of congestion pricing for parking spaces in the city of San Francisco and show that the welfare is increased

in popular areas but reduced in less-congested areas. For airline tickets, the pricing practice can be classified into dynamic pricing, which arises from demand uncertainty and is the main subject of our paper, and intertemporal price discrimination. Dana (1999) and Lazarev (2013) find that social welfare can either increase or decrease with intertemporal price discrimination, and that business travelers (with a high disutility of waiting) can benefit from an increasing number of tourists (with a low disutility of waiting). Dana (1998) investigates the effect of advanced-purchase discounts (which can be interpreted as a form of dynamic pricing) when demand is uncertain, and concludes that welfare is decreased when high-valuation customers have more uncertain demand but unchanged when low-valuation customers have more uncertain demand. Our result differs from his because (1) our model does not consider the heterogeneity in demand uncertainty of customers having different valuations, and (2) the dynamic nature of our model mitigates the "misallocation effect" defined in Dana (1998) that is caused by rationing. Williams (2013) empirically shows that travelers benefit from dynamic pricing using a data set of daily fares at the flight level. McAfee and Velde (2008) investigate dynamic pricing with constant demand elasticity and prove that the revenue-maximizing pricing policy can also achieve efficiency. The insights drawn in this paper strengthen their point by helping to explain why and for what demand functions this is likely to happen. Although the implications of our results (that dynamic pricing can benefit consumers) are similar to those of some of the above papers, the main differences are as follows: (1) We study the welfare-maximizing pricing policy based on the model by Gallego and van Ryzin (1994) and show that when the system is scaled, the welfare-maximizing pricing policy is similar to the revenue-maximizing pricing policy. (2) We explain why dynamic pricing is usually win–win compared to the revenue-maximizing static price, and derive conditions of the demand function under which dynamic pricing is likely to benefit consumers.

## 2. Dynamic Pricing to Maximize Social Welfare

In this section, we review the concept of consumers' surplus and consider a central planner who attempts to maximize social welfare when a firm has a finite horizon to sell a given inventory of perishable, non-replenishable products. Structural properties of the welfare-maximizing pricing policy are explored. We then conduct an asymptotic analysis in the fluid regime following the methodology of Gallego and van Ryzin (1994). We show that when capacity is scarce, the same market-clearing fixed-price heuristic that asymptotically maximizes revenue also asymptotically maximizes welfare.

**Table 1.** Constant Elasticity Demand (CES), Exponential Demand, and Linear Demand

	$d(p)$	$\mathcal{S}(p)$	$U(q)$	Parameters
CES	$ap^{-b}$	$\frac{a}{b-1}p^{1-b}$	$\frac{b}{b-1}a^{1/b}q^{1-1/b}$	$a > 0, b > 1$
Exponential	$\lambda \exp(-p/\theta)$	$\lambda \theta \exp(-p/\theta)$	$\theta q(1 + \ln \lambda - \ln q)1_{q \leq \lambda} + \theta \lambda 1_{q > \lambda}$	$\theta, \lambda > 0$
Linear	$(b-p)^+ / 2a$	$((b-p)^+)^2 / 4a$	$(-aq^2 + bq)1_{q \leq b/2a} + b^2 / 4a 1_{q > b/2a}$	$a, b > 0$

**2.1. Consumers’ Surplus**

Suppose the firm sets a price of  $p$  and there is a representative consumer (see, e.g., Varian 2010, Anderson et al. 1988) who acts on behalf of all consumers in the market. For  $q$  units of the product, the representative consumer’s monetary utility  $U(q)$  of the product is an increasing, concave, and twice differentiable function with  $U(0) = 0$ . The maximum net utility is given by  $\mathcal{S}(p) \triangleq \max_{q \geq 0} \{U(q) - pq\}$ . To maximize her net utility, the representative consumer chooses the quantity  $q = d(p) \triangleq (U')^{-1}(p)$  obtained by the first-order necessary condition, which is also sufficient from the assumed concavity of  $U$ . Here,  $(\cdot)^{-1}$  denotes the inverse function. This optimization defines consumers’ surplus  $\mathcal{S}(\cdot)$  and market demand  $d(\cdot)$ . By the envelope theorem, the surplus of the representative consumer satisfies  $\mathcal{S}'(p) = -d(p)$ . Moreover, since consumers earn no surplus at  $p = \infty$ , we have  $\mathcal{S}(p) = \int_p^\infty d(x) dx$ .

In Table 1, we provide three instances of the utility function  $U(\cdot)$ , the demand function  $d(\cdot)$ , and the consumers’ surplus  $\mathcal{S}(\cdot)$  in closed forms. Another way to interpret the relationship between  $d(p)$  and  $\mathcal{S}(p)$  without invoking the representative consumer and the utility function is to assume a mass  $\lambda$  of nonatomic consumers with random WtP  $\Omega$ . The market demand is  $d(p) = \lambda P(\Omega \geq p)$ , as only those customers with WtP higher than  $p$  buy the product. Their surplus is the difference, if positive, between the WtP and the price  $\lambda E[(\Omega - p)^+] = \lambda \int_p^\infty P(\Omega \geq x) dx = \int_p^\infty d(x) dx = \mathcal{S}(p)$ . Both interpretations are consistent, and we will use them interchangeably in the paper.

**2.2. Optimal Social Welfare: Stochastic Case**

The issue of capacity constraints with random demand arises naturally for firms aiming to maximize revenue, such as airlines, hotels, concert venues, and car rental agencies. In these settings, the firm has a fixed investment in finite capacity  $c$  that is not replenished during the sales horizon  $[0, T]$ . For convenience, we denote  $t \in [0, T]$  as the time-to-go, and  $x$  as the remaining inventory. At time  $t = T$ , the firm has  $x = c$  units of capacity. Demand is assumed to be a Poisson process with intensity  $d(p)$  for price  $p$ . We assume without loss of generality that unsold items at  $t = 0$  have zero salvage value. The firm’s revenue-maximizing dynamic pricing in this setting is studied in Gallego

and van Ryzin (1994). See Elmaghraby and Keskinocak (2003) for a comprehensive survey.

To provide the intuition for the formulation of the welfare optimization problem, we first summarize the results for the revenue-maximizing dynamic pricing policy. Let  $V(t, x)$  be the optimal expected revenue of the firm for time-to-go  $t \geq 0$  and  $x$  units of inventory. It can be shown that  $V(t, x)$  satisfies

$$\frac{\partial V(t, x)}{\partial t} = \max_{p \geq 0} \{d(p)(p - \Delta V(t, x))\}, \tag{1}$$

$$V(0, x) = V(t, 0) = 0,$$

where  $\Delta V(t, x) \triangleq V(t, x) - V(t, x - 1)$  represents the marginal value of the  $x$ th unit of inventory. The revenue-maximizing (dynamic) price for state  $(t, x)$  is denoted  $p_V^*(t, x) = p(\Delta V(t, x)) \triangleq \arg \max_{p \geq 0} \{d(p)(p - \Delta V(t, x))\}$ . Under mild conditions,<sup>1</sup>  $p_V^*(t, x)$  exists and is increasing in  $t$  and decreasing in  $x$ .

Next, we consider the optimal expected social welfare  $W(t, x)$  generated over the sales horizon for a nonanticipating pricing policy  $p(t, x)$ . We will show that the problem essentially adds the consumer’s surplus conditioning on sales to the price in the objective function corresponding to the firm’s problem. Indeed, at price  $p$ , the firm earns revenue at rate  $pd(p)$ , and consumers earn surplus at rate  $\mathcal{S}(p)$ ; hence, social welfare increases at rate  $(p + \mathcal{S}(p)/d(p))d(p)$ . To maximize social welfare  $W(t, x)$ , the central planner uses  $p_W^*(t, x)$  for state  $(t, x)$ .

**Proposition 1.** *The optimal social welfare  $W(t, x)$  satisfies*

$$\frac{\partial W(t, x)}{\partial t} = \max_p \{\mathcal{S}(p) + d(p)(p - \Delta W(t, x))\}$$

$$= \mathcal{S}(\Delta W(t, x)), \quad x \geq 1, t \geq 0, \tag{2}$$

$$W(0, x) = W(t, 0) = 0,$$

where  $\Delta W(t, x) = W(t, x) - W(t, x - 1)$  represents the marginal value of capacity for social welfare. Moreover,  $p_W^*(t, x) = \Delta W(t, x)$  is increasing in  $t$  and decreasing in  $x$ .

**Remark 1.** (1) The key element of the equations for both revenue and social welfare is the “shadow cost” of selling one unit of the product. For the revenue-maximizing problem, the shadow cost is the marginal value of capacity for revenue  $\Delta V$ ; in the welfare problem, it is the marginal value of capacity for welfare  $\Delta W$ . The optimal revenue rate  $\partial V / \partial t$  is therefore

$\max_p \{d(p)(p - \Delta V)\}$ , as if the firm were maximizing the profit of selling one unit of its product at variable cost  $\Delta V$ . The optimal social welfare rate  $\partial W / \partial t$  is equal to  $\mathcal{S}(\Delta W)$ , as if the central planner were maximizing the social welfare of selling one unit of the product at cost  $\Delta W$ ,  $\mathcal{S}(p) + d(p)(p - \Delta W)$ .

(2) Recall the static pricing problem: if the cost is  $z$ , then a price  $p$  results in profit  $d(p)(p - z)$  and consumers' surplus  $\mathcal{S}(p)$ . It can be shown that to maximize welfare in the static case, the price should be set equal to the marginal cost  $z$ , and all of the welfare is earned by customers (see the proof of Proposition 1 for the proof for this claim). In contrast, in the dynamic case, the price is equal to the marginal value of capacity for welfare, which is positive in general; thus, the firm can gain a positive fraction of the total social welfare.

(3) The welfare-maximizing price is not necessarily lower than the revenue-maximizing price (see Section 4 for an example), although this is true in most cases.

(4) Although  $p_w^*(t, x)$  is not equal to  $p_v^*(t, x)$ , they share the same structural properties—i.e., they both increase in  $t$  and decrease in  $x$ . This implies that to maximize social welfare, the central planner should also immediately increase the price when an item is sold, and gradually decrease the price over time, exactly as the firm would do.

### 2.3. Optimal Social Welfare: Fluid Approximation

The fluid approximation has been studied extensively in the literature of dynamic pricing. The Poisson arrival process of rate  $d(p)$  is replaced by a deterministic demand flow equal to the mean rate: within an infinitesimal time period  $dt$ , the size of the demand is  $d(p)dt$  for price  $p$ . The revenue rate is therefore  $pd(p)$ , and the surplus rate is  $\mathcal{S}(p)$ . This approximation usually provides tractable heuristic pricing policies that perform well in large-scale systems (see Section 2.4 for details). In this section, we will demonstrate that the pricing policies that maximize welfare, revenue, and consumers' surplus, respectively, in the fluid approximation actually coincide in many cases.

First consider the following welfare-maximizing problem:

$$\begin{aligned} \bar{W}(T, c) = \max_{p_t, 0 \leq t \leq T} & \int_0^T (\mathcal{S}(p_t) + p_t d(p_t)) dt \\ \text{subject to} & \int_0^T d(p_t) dt \leq c. \end{aligned} \quad (3)$$

The objective is the total welfare over the horizon of the fluid approximation. The constraint states that the total demand should be less than the initial capacity. Proposition 2 shows that when the initial capacity is scarce, then the market-clearing price  $p^{mc} \triangleq d^{-1}(c/T)$  is welfare maximizing.

**Proposition 2.** *If  $d(p)$  is continuous and decreasing, and  $c < Td(0)$ , then the optimal solution to (3) is  $p_t \equiv p^{mc}$  for all  $t \in [0, T]$ . Moreover,  $\bar{W}(T, c) \geq W(T, c)$ .*

We now consider the fluid problems for the firm and consumers. For the firm (see Gallego and van Ryzin 1994 for details), the revenue-maximizing problem is

$$\begin{aligned} \bar{R}(T, c) = \max_{p_t, 0 \leq t \leq T} & \int_0^T p_t d(p_t) dt \\ \text{subject to} & \int_0^T d(p_t) dt \leq c. \end{aligned} \quad (4)$$

The optimal solution is to set the price at  $p_t \equiv \max\{p^{mc}, \hat{p}\}$  for the entire horizon, where

$$\hat{p} \triangleq \arg \max_{p \geq 0} \{pd(p)\}$$

is the revenue-maximizing price when capacity is unconstrained. The solution is  $p_t = p^{mc}$  when  $c \leq Td(\hat{p})$ . This is the case when capacity is insufficient to satisfy demand at the unconstrained revenue maximizer  $\hat{p}$ . It is the most relevant case in revenue management applications.

The surplus-maximizing fluid problem for consumers is given by

$$\begin{aligned} \bar{S}(T, c) = \max_{p_t, 0 \leq t \leq T} & \int_0^T \mathcal{S}(p_t) dt \\ \text{subject to} & \int_0^T d(p_t) dt \leq c. \end{aligned} \quad (5)$$

When  $c \geq Td(0)$ , the price that maximizes surplus is  $p_f \equiv 0$ . This is clear since  $p_f = 0$  is a feasible solution and maximizes the objective function pointwise.

The problem is more subtle when capacity is scarce  $c < Td(0)$ , and some definitions are required before tackling this problem. Assume that  $d(\cdot)$  is everywhere differentiable and let  $h(p) \triangleq -d'(p)/d(p)$  be the hazard rate associated with the random variable  $\Omega$  with tail probability  $P(\Omega \geq p) = d(p)/d(0)$  when  $d(0) < \infty$ . When  $\Omega$  is well defined,  $\mathcal{S}(p)/d(p) = E[\Omega - p | \Omega \geq p]$  is the expected conditional mean residual life at  $p$ . Recall that the random variable  $\Omega$  is NBUE (new better than used in expectation) if  $\mathcal{S}(p)/d(p)$  is decreasing in  $p$ . Moreover, an increasing  $h(p)$  implies that the associated random variable is NBUE (see Hollander and Proschan 1972 and references therein). We find that the distribution of consumers' WtP  $\Omega$  plays an important role in determining the form of the surplus-maximizing pricing policy. We are now ready to address the surplus-maximizing problem when capacity is scarce.

**Proposition 3.** *Suppose  $c < Td(0)$ . If  $\Omega$  is strictly NBUE, then the optimal solution to (5) is  $p_t \equiv 0$  until the inventory is depleted. If  $h(p)$  is strictly decreasing, then  $p_t \equiv p^{mc}$ .*

Proposition 3 states that if  $\Omega$  is strictly NBUE, which is guaranteed if  $h(p)$  is strictly increasing, then the firm prices its product at  $p_t = 0$  until capacity is exhausted to maximize consumers' surplus. The firm then uses the choke price, possibly infinity, to turn off demand, generating no additional surplus during the rest of the horizon. This scheme results in surplus  $c\mathcal{S}(0)/d(0)$ . In contrast, if  $h(p)$  is strictly decreasing, then the market-clearing price maximizes consumers' surplus, resulting in surplus  $T\mathcal{S}(p^{mc}) = c\mathcal{S}(p^{mc})/d(p^{mc})$ . We now present three examples: the CES demand has a decreasing hazard rate, and  $p^{mc}$  maximizes (5); the exponential demand has a constant hazard rate, and any price between  $[0, p^{mc}]$  is surplus maximizing; the linear demand function has an increasing hazard rate, and thus  $p_t = 0$  maximizes consumers' surplus.

One may be surprised that a low price does not always benefit consumers. This is because the firm only has a fixed initial capacity. When the price is low, the demand is high and the inventory is depleted quickly. However, it is not guaranteed that the customers who buy the product have high WtP. In some cases (when the hazard rate is decreasing), it is beneficial to raise the price to "screen out" consumers with low WtP; the increase in the average WtP of customers who buy the product is higher than the increase in price. The total consumers' surplus may increase as a result.

Combining the previous results with Proposition 3, we see that if  $c < Td(\hat{p})$ , then the welfare  $\bar{W}(T, c)$  is decomposed into two components: the firm's revenue  $cp^{mc}$  and the consumers' surplus  $T\mathcal{S}(p^{mc})$ , with the market-clearing price maximizing both welfare and revenue. In addition, if  $h(p)$  is decreasing, then the market-clearing price also maximizes consumers' surplus. On the other hand, if  $\Omega$  is NBUE, then using the market-clearing price results in a loss to consumers compared to the optimal consumers' surplus. The gap is equal to  $c\mathcal{S}(0)/d(0) - T\mathcal{S}(p^{mc}) = c(\mathbb{E}[\Omega] - \mathbb{E}[\Omega - p^{mc} | \Omega > p^{mc}])$ . These results for the fluid approximation suggest that the market-clearing price may be a good heuristic for revenue and welfare when capacity is scarce, and also a reasonable heuristic for surplus. This finding is summarized below:

**Corollary 1.** *If  $c \leq Td(\hat{p})$  and  $h(p)$  is decreasing, then  $\bar{W}(T, c) = \bar{R}(T, c) + \bar{S}(T, c)$ , and the market-clearing price simultaneously optimizes the firm's revenues, social welfare, and consumers' surplus in the fluid approximation.*

### 2.4. Asymptotic Optimality of the Fluid Solution

The fluid approximation is closely related to the following asymptotic regime of the stochastic system when we use the WtP interpretation  $d(p) = \lambda P(\Omega \geq p)$ : if the initial capacity  $c$  and the aggregate arrival rate  $\lambda T$  tend to infinity while the ratio  $\lambda T/c$  remains constant, then the Poisson arrival process can be approximated using the fluid approximation. Gallego and van Ryzin (1994) have shown that the fixed price  $p_f = \max\{p^{mc}, \hat{p}\}$

is asymptotically optimal for the firm. More precisely, these authors show that  $V(T, c | p_f)/V(T, c) \rightarrow 1$ , where  $V(T, c | p_f)$  is the expected revenue when a fixed price  $p_f$  is used until the end of the sales horizon or until capacity is exhausted (in which case the choke price is used). This is a highly relevant regime because demand and capacity are both high in many industries.

Next, we show that when capacity is scarce (i.e.,  $c \leq Td(\hat{p})$ ), the fixed price  $p_f = p^{mc}$  is asymptotically optimal for social welfare as well. We consider a sequence of systems indexed by  $n$ :  $c^{(n)} = nc$  and  $\lambda^{(n)}T^{(n)} = n\lambda T$ . Note that the values of  $\lambda$  and  $T$  do not essentially change the pricing policy (up to a linear change of state variables) or the value function, as long as their product  $\lambda T$  is fixed.

**Proposition 4.** *If  $c < Td(\hat{p})$ , then the welfare generated by the market-clearing price  $p^{mc}$  is asymptotically optimal—i.e.,*

$$\lim_{n \rightarrow \infty} \frac{W(T^{(n)}, c^{(n)} | p^{mc})}{W(T^{(n)}, c^{(n)})} = 1.$$

Proposition 4 states that for large-scale scarce-capacity systems, the central planner can simply use the market-clearing price and generate welfare that is  $o(n)$  less than the optimal social welfare. Meanwhile, the firm also accepts the policy because it allows it to achieve revenue that is only  $o(n)$  less than the optimal revenue.

By Proposition 3, the market-clearing price also asymptotically maximizes consumers' surplus when the hazard rate is decreasing. Therefore, for large-scale systems, using the market-clearing price is win-win to a large extent, because it pleases both the firm and customers, and thus also the central planner.

### 3. Pareto Efficiency

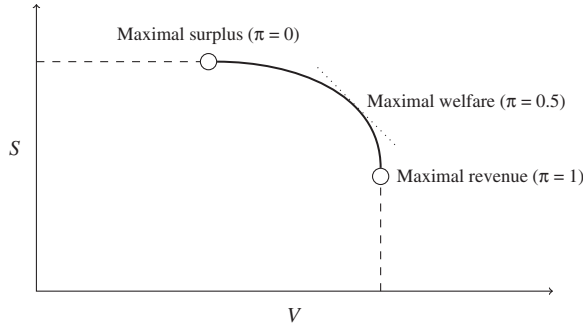
Both the welfare-maximizing and revenue-maximizing pricing policies considered in Section 2 are "Pareto efficient" in the following sense:

**Definition 1.** A nonanticipating pricing policy is Pareto efficient if no other nonanticipating pricing policies can weakly improve both the expected revenue and consumers' surplus and strictly improve at least one of them.

Indeed, under mild conditions, the welfare-maximizing and revenue-maximizing pricing policies are unique, and no other policies can weakly improve social welfare or revenues, respectively.

In this section, we characterize all Pareto-efficient pricing policies and the corresponding welfare allocations. Because of the Markov property of the system, the family of nonanticipating pricing policies can be expressed in the form of  $p(t, x)$ —i.e., the price is only

**Figure 1.** Pareto Frontier of the Allocation of Revenue and Consumers' Surplus for a Given Initial State  $(T, c)$



a function of the current time-to-go and the remaining inventory. For any  $\pi \in [0, 1]$ , consider the problem of maximizing a convex combination of the objective function of the firm and the objective function of the consumers. More precisely, assume that the quantity of interest is  $\pi \times$  revenue  $+ (1 - \pi) \times$  consumers' surplus. Let  $W_\pi(T, c)$  denote its optimal value, which satisfies a similar differential equation to (2) whereby we replace  $d(p)(p + \mathcal{S}(p)/d(p))$  by  $d(p)(\pi p + (1 - \pi)\mathcal{S}(p)/d(p))$ . The pricing policy that achieves the optimal value is denoted as  $p_\pi(t, x)$ . Under this policy, the expected revenue and consumers' surplus are denoted as  $V_\pi$  and  $S_\pi$ , respectively.

**Proposition 5.** A pricing policy is Pareto efficient if and only if it is  $p_\pi$  for some  $\pi \in [0, 1]$ .

When  $\pi = 1$ , the firm's revenue is maximized; when  $\pi = 0.5$ , social welfare is maximized; when  $\pi = 0$ , consumers' surplus is maximized. As  $\pi$  ranges from 0 to 1,  $(V_\pi, S_\pi)$  generates the Pareto frontier on the  $V - S$  plane, as illustrated in Figure 1. By definition, the Pareto frontier is always concave; when a straight line of slope  $-\pi/(1 - \pi)$  is drawn, its tangent point to the frontier corresponds to  $W_\pi(T, c)$ . In particular, the tangent point of the vertical line maximizes revenue (i.e.,  $\pi = 1$ ). For an arbitrary nonanticipating pricing policy, the generated consumers' surplus and revenue correspond to a point inside the area enclosed by the Pareto frontier, the dotted lines, and the axes.

### 3.1. Asymptotic Analysis of the Pareto Frontier

We already know that dynamic pricing  $p_V^*(t, x)$  is the optimal nonanticipating policy for the firm. We will now show that it also asymptotically maximizes social welfare. In fact, we will present a stronger and deeper result: the Pareto frontier "shrinks" relative to  $n$  in the same regime as in Section 2.4. This result shows that the zero-sum hypothesis is false: even when the firm is only maximizing its revenue using dynamic pricing, the resulting social welfare is almost as substantial as that generated by the welfare-maximizing dynamic pricing policy. This means that even in the

rare case that consumers are hurt by dynamic pricing, the firm's gain exceeds the loss to consumers. Consider the asymptotic regime  $c^{(n)} = nc$  and  $\lambda^{(n)}T^{(n)} = n\lambda T$ .

**Theorem 1.** If  $c \leq Td(\hat{p})$  and  $g(x) \triangleq xd^{-1}(x)$  is concave for  $x \leq d(\hat{p})$ , then the length of the segment  $\pi \in [0.5, 1]$  divided by  $n$  converges to 0 as  $n \rightarrow \infty$ .

Theorem 1 states that in the limit, the segment  $\pi \in [0.5, 1]$  grows more slowly than  $n$ . Since both social welfare and revenues grow linearly in  $n$ , the distance between the two allocations  $\pi = 1$  and  $\pi = 0.5$  becomes negligible when  $n$  is large. This implies that any Pareto-efficient pricing policy  $p_\pi$  for  $\pi \in [0.5, 1]$ , including  $p_{\pi=1} = p_V^*$  and  $p_{\pi=0.5} = p_W^*$ , is asymptotically optimal for all  $\pi \in [0.5, 1]$ . Theorem 1 has significant policy and practice implications. For policy makers concerned about the social welfare generated by firms that use dynamic pricing, our results show that dynamic pricing does not hurt welfare significantly, especially for industries with large-scale demand and capacity. It is worth pointing out that when  $c > Td(\hat{p})$ , the segment  $\pi \in [0.5, 1]$  grows linearly in  $n$ . Moreover, the other half segment,  $\pi \in [0, 0.5]$ , does not shrink in  $n$  when the hazard rate is increasing, as implied by Proposition 3. Thus, neither the market-clearing price nor the dynamic policy  $p_V^*(t, x)$  can asymptotically maximize consumers' surplus in that case.

### 3.2. A Numerical Example

In this section, we provide numerical comparisons of  $p_V^*$  and  $p_W^*$ . We showed in Section 2.4 that  $p_V^*$  tends to be close to  $p_W^*$  when capacity is constrained in large-scale systems. The objective of this section is to illustrate that the revenue-maximizing and welfare-maximizing pricing policies are similar even for small-scale systems. Consider exponential demand  $d(p) = \exp(-p)$  and  $\mathcal{S}(p) = \exp(-p)$ . Table 2 shows the ratio of  $p_V^*(T, c)/p_W^*(T, c)$  for different values of  $T$  and  $c$ . When  $c/T$  is small (the lower triangle),  $p_V^*$  is already very close to  $p_W^*$  even for small-scale systems such as  $c = 1$  and  $T = 20$ . As capacity becomes increasingly scarce, the revenue-maximizing price approaches the welfare-maximizing price. When  $c/T$  is fixed (on the diagonal),

**Table 2.** Ratio of  $p_V^*(T, c)/p_W^*(T, c)$  for Different  $T$  and  $c$

T	c					
	1	5	10	20	50	100
4	<b>1.18</b>	4.56	18.79	$1 \times 10^8$	$\infty$	$\infty$
20	1.02	<b>1.10</b>	1.42	5.78	$1 \times 10^8$	$\infty$
40	1.01	1.03	<b>1.07</b>	1.41	53.00	$1 \times 10^{15}$
80	1.00	1.01	1.01	<b>1.04</b>	2.00	249.99
200	1.00	1.00	1.00	1.00	<b>1.02</b>	1.42
400	1.00	1.00	1.00	1.00	1.00	<b>1.01</b>

Note. In the lower triangle, the ratio is very close to 1.



the ratio converges to one as  $c$  increases. Because exponential demand functions have constant hazard rates, Corollary 1 implies that  $\bar{W} = \bar{R} + \bar{S}$  and  $p_V^*$  also asymptotically achieves the optimal consumers' surplus.

#### 4. When Is Dynamic Pricing Win–Win?

For unregulated industries, firms with monopoly power usually use either the revenue-maximizing dynamic pricing policy  $p_V^*(t, x)$  or the revenue-maximizing static price  $p_f^*$ . If firms are concerned about their reputation and future revenues given the negative public perception of dynamic pricing, then they may choose  $p_f^*$  and forgo the benefits of dynamic pricing. In this section, we will clarify this misconception and show that dynamic pricing is in many cases win–win compared to using the static price  $p_f^*$ . Moreover, we will identify conditions of the WtP distribution under which dynamic pricing is likely to be win–win.

##### 4.1. Pricing and Consumers' Surplus

We first derive the consumers' surplus for an arbitrary static price  $p_f$  and the revenue-maximizing dynamic pricing policy  $p_V^*(t, x)$ . We showed in Section 2.2 that the generated surplus conditioning on a sale given price  $p$  is  $\mathcal{S}(p)/d(p)$ . Thus, by replacing the revenue  $p$  with  $\mathcal{S}(p)/d(p)$  in (1), we obtain the differential equation for the consumers' surplus  $S(t, x | p)$  generated by any nonanticipating pricing policy  $p(t, x)$ . In particular, for the revenue-maximizing pricing policy  $p_V^*$ , the consumers' surplus for time-to-go  $t$  and remaining inventory  $x$  is  $S(t, x | p_V^*)$ . For any static price  $p_f$ ,  $S(t, x | p_f) = E[\mathcal{S}(p_f)/d(p_f) \times \min\{c, \text{Pois}(Td(p_f))\}]$ , where  $\text{Pois}(\lambda)$  represents a Poisson random variable with mean  $\lambda$ . We will compare  $S(t, x | p_V^*)$  and  $S(t, x | p_f^*)$ , where  $p_f^* = \arg \max_p \{p E[\min\{c, \text{Pois}(Td(p))\}]\}$  is the revenue-maximizing static price. It is worth mentioning that  $p_f^*$  remains a function of the initial state  $(T, c)$ , while being fixed over the horizon.

We first investigate the Pareto efficiency of the revenue-maximizing static price. We will show that a static policy is rarely Pareto efficient—i.e., it usually results in a welfare allocation that is strictly inside the Pareto frontier. This implies that we can find a pricing policy (not necessarily  $p_V^*$ ) that improves both revenue and consumers' surplus over the revenue-maximizing static price.

**Proposition 6.** *The revenue-maximizing static price  $p_f^*$  results in a welfare allocation that lies on the Pareto frontier only if both of the following conditions are satisfied:*

1. It equals to the unconstrained revenue maximizer  $\hat{p}$ ;
2.  $pd(p)$  is not differentiable at  $p_f^*$ —i.e.,

$$\lim_{p \rightarrow (p_f^*)^-} \frac{p(d(p)) - p_f^* d(p_f^*)}{p - p_f^*} > \lim_{p \rightarrow (p_f^*)^+} \frac{p(d(p)) - p_f^* d(p_f^*)}{p - p_f^*}.$$

The first condition only holds when the initial capacity is abundant relative to demand at  $\hat{p}$ —i.e.,  $c > Td(\hat{p})$ .

However, it rarely holds in revenue management applications as the theory of dynamic pricing mainly focuses on scenarios of scarce capacity. The second condition implies that the function  $pd(p)$  “sticks out” at  $p_f^*$ . Those two conditions are rarely satisfied in practice; hence, the revenue-maximizing static price is almost never Pareto efficient.

##### 4.2. Common Demand Functions

Next, we study the expected consumers' surplus generated by the revenue-maximizing dynamic/static pricing policies for three common demand functions: constant elasticity demand, exponential demand, and linear demand. We show that consumers are better off with dynamic pricing in most cases.

**Constant elasticity demand:** Consider  $d(p) = ap^{-b}$  and thus  $\mathcal{S}(p) = ap^{1-b}/(b-1)$ , for  $a > 0$  and  $b > 1$ . It is first studied in Karlin and Carr (1962) and has become a popular model to characterize how demand reacts to changes in price. The advantage of the constant elasticity demand is that it can be transformed into a linear function by taking logarithms, making the parameters relatively easy to estimate. Its other advantages include its ability to characterize the nonlinear effects arising in the market and the possibility of deriving it from the Cobb–Douglas production function.

Recall that  $V(t, x | p)$  denotes the revenue earned by the firm for a nonanticipating pricing policy  $p$ . The elegant connection between the revenue and consumers' surplus of the CES demand is first discovered in McAfee and Velde (2008) and is stated in a more general form here:

**Proposition 7.** *If  $d(p) = ap^{-b}$  for  $a > 0$  and  $b > 1$ , then for any nonanticipating pricing policy (including static policies), we have*

$$V(t, x | p) = (b-1)S(t, x | p).$$

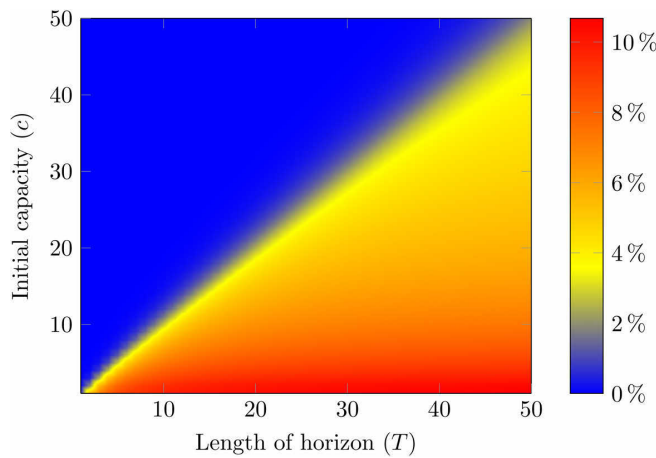
Proposition 7 states that for any initial states, the firm's revenue is always a constant multiple of the consumers' surplus for any nonanticipating pricing policies. This implies that

$$\frac{S(t, x | p_V^*)}{S(t, x | p)} = \frac{V(t, x | p_V^*)}{V(t, x | p)} \geq 1,$$

because  $p_V^*$  maximizes the firm's revenue. In other words, the revenue-maximizing dynamic pricing policy also maximizes consumers' surplus. Moreover, compared to any static pricing policy, both the firm and consumers benefit from dynamic pricing at the same rate. The Pareto frontier of the CES demand function collapses to a single point. All Pareto-efficient allocations correspond to the same pricing policy—i.e.,  $p_V^*(t, x)$ , for all  $\pi \in [0, 1]$ . In particular,  $p_V^*(t, x) = p_W^*(t, x)$ .

We conduct a numerical experiment to investigate the percentage improvement in consumers' surplus

**Figure 2.** (Color online) Percentage Improvement Due to Dynamic Pricing in the Expected Surplus (Revenue) Over the Optimal Static Pricing Policy, for  $d(p) = ap^{-b}$ ,  $\mathcal{S}(p) = ap^{1-b}/(b-1)$ , and  $a = 1, b = 2$



using the dynamic pricing policy over the revenue-maximizing static price—i.e.,  $(S(T, c | p_V^*) - S(T, c | p_f^*)) / S(T, c | p_f^*)$  or equivalently  $(V(T, c | p_V^*) - V(T, c | p_f^*)) / V(T, c | p_f^*)$ . Figure 2 illustrates the results for different initial states  $(T, c)$ , ranging from scarce to abundant capacity. It suggests that dynamic pricing is especially effective when capacity is scarce ( $c$  is small relative to  $T$ ).

**Exponential demand:** Consider  $d(p) = \lambda \exp(-p/\theta)$  for  $\lambda, \theta > 0$  and  $\mathcal{S}(p) = \lambda \theta \exp(-p/\theta)$ . Exponential demand is used extensively in the literature on marketing and revenue management, because it is well defined when  $p \rightarrow 0$  and infinitely differentiable. It allows for an elegant and insightful solution to  $V(t, x | p_V^*)$  (Gallego and van Ryzin 1994). This analytical tractability is preserved for consumers' surplus.

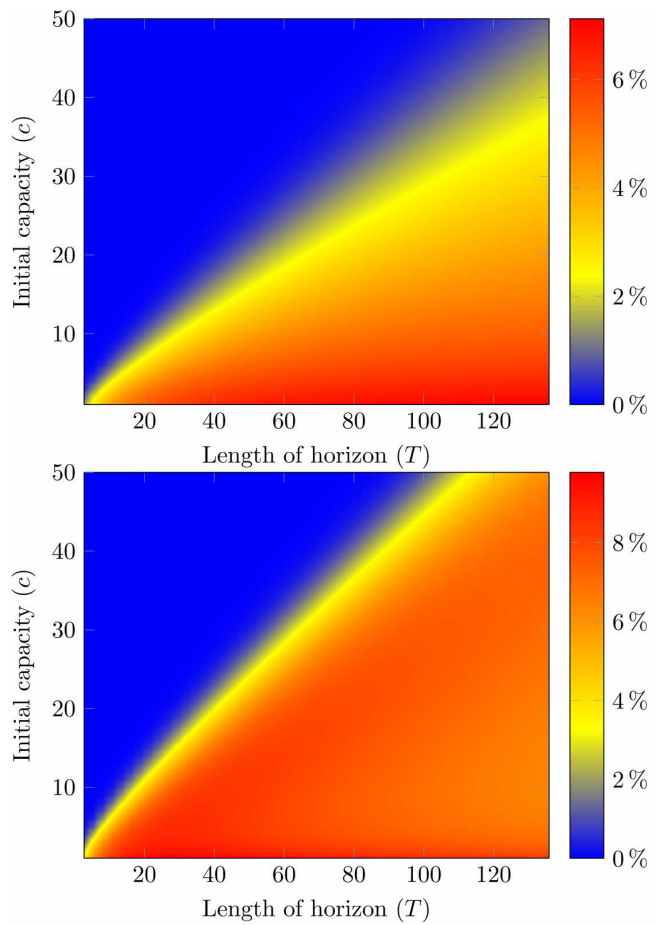
**Proposition 8.** Consider  $d(p) = \lambda \exp(-p/\theta)$ ,  $\mathcal{S}(p) = \lambda \theta \exp(-p/\theta)$ . Let  $\Lambda = \lambda t/e$ . We have

$$\begin{aligned} V(t, x | p_V^*) &= \theta(\Lambda + \log(\mathbb{P}(\text{Pois}(\Lambda) \leq x))), \\ p_V^*(t, x) &= \theta + \theta \log\left(\frac{\mathbb{P}(\text{Pois}(\Lambda) \leq x)}{\mathbb{P}(\text{Pois}(\Lambda) \leq x - 1)}\right), \\ S(t, x | p_V^*) &= \theta \mathbb{E}[\text{Pois}(\Lambda) | \text{Pois}(\Lambda) \leq x], \\ V(t, x | p_f) &= p_f \mathbb{E}[\min\{\text{Pois}(td(p_f)), x\}], \\ S(t, x | p_f) &= \theta \mathbb{E}[\min\{\text{Pois}(\lambda td(p_f)), x\}]. \end{aligned}$$

The solutions can be directly verified from the differential equations.

Although  $S(t, x | p_V^*)$  has a closed form, we cannot solve the revenue-maximizing static price explicitly. Thus, we resort to the numerical computation of  $p_f^*(T, c)$ . Given that the relevant quantities only depend on  $\lambda t$  and  $x$ , we can simply set  $\theta = \lambda = 1$ , and the numerical result reflects all possible parameter values. As illustrated in Figure 3, dynamic pricing always benefits consumers, especially when capacity is scarce.

**Figure 3.** (Color online) Percentage Improvements Due to Dynamic Pricing in the Expected Revenue (Upper Panel) and Consumers' Surplus (Lower Panel) Over the Optimal Static Pricing Policy, for  $d(p) = \lambda \exp(-p/\theta)$  and  $\mathcal{S}(p) = \lambda \theta \exp(-p/\theta)$



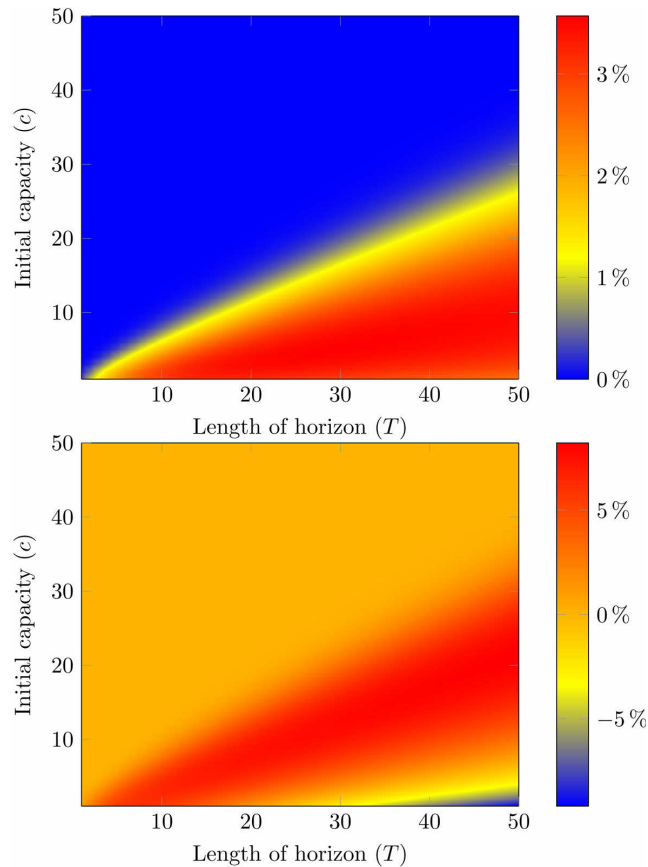
**Linear demand:**  $d(p) = \lambda(1 - ap)^+$  is another commonly used demand function because of its simplicity and analytical tractability. Consumers of linear demand have quadratic surplus  $\mathcal{S}(p) = \lambda((1 - ap)^2)/2a$ . Unfortunately, in the context of dynamic pricing, there are few analytical results for the revenue or consumers' surplus. The numerical results are illustrated in Figure 4 for  $\lambda = a = 1$ . For many initial state  $(T, c)$ , consumers benefit from dynamic pricing—i.e.,  $S(T, c | p_V^*) \geq S(T, c | p_f)$ . However, dynamic pricing hurts consumers when capacity is extremely scarce—i.e.,  $T$  is very large and  $c$  is very small.

To investigate this phenomenon, consider the asymptotic regime  $c = 1$  and  $T \rightarrow \infty$ . For simplicity, let  $d(p) = 1 - p$ . By solving the differential equations explicitly, for the dynamic pricing policy, we have

$$p_V^*(T, 1) = \frac{2 + T}{4 + T}, \quad S(T, 1 | p_V^*) = \frac{2T}{(T + 4)^2}.$$

On the other hand, the optimal static price maximizes  $p \mathbb{E}[\min\{1, \text{Pois}(T(1 - p))\}] = p(1 - \exp(-T(1 - p)))$

**Figure 4.** (Color online) Percentage Improvements Due to Dynamic Pricing in the Expected Revenue (Upper Panel) and Consumers' Surplus (Lower Panel) Over the Optimal Static Pricing Policy, for  $d(p) = \lambda(1 - ap)^+$  and  $\mathcal{S}(p) = \lambda((1 - ap)^+)^2/2a$



and thus  $p_f^* = 1 - O(\log(T)/T)$ . Hence, the consumers' surplus for the optimal static price is approximately  $S(T, 1|p_f^*) = (1 - p_f^*)(1 - \exp(-T(1 - p_f^*))) / 2 = O(\log(T)/T)$ , dominating  $S(T, 1|p_v^*) = O(1/T)$ . Nevertheless, it can be shown that the social welfare generated by  $p_v^*$ , which is  $1 - O(1/T)$ , is higher than that of  $p_f^*$ , which is  $1 - O(\log(T)/T)$ , in the same asymptotic regime.

For general demand functions, the revenue-maximizing static/dynamic prices usually do not have explicit forms. Numerical computations may therefore be necessary. Nevertheless, we attempt to partially answer this question for general demand functions in Section 4.4.

### 4.3. Discussion

Among the three examples, revenue-maximizing dynamic pricing is win-win for the CES and exponential demand functions. For CES demand, it even generates more consumers' surplus than any static pricing policy. For linear demand, it may hurt consumers' surplus when the initial capacity is extremely scarce relative to the sales horizon. Two natural questions arise: Why does dynamic pricing improve consumers' surplus for

most distributions of customers' WtP but not for others? Which characteristics of a demand function make dynamic pricing favorable to customers?

We provide a heuristic answer to the first question here while formalizing it and answering the second question in Section 4.4. The intuition lies in how the *welfare* is allocated between the firm and the customers *conditioning on a sale*—i.e., the tuple  $(p, \mathcal{S}(p)/d(p))$ . Recall that if the price is  $p$ , then on a sale, the firm earns revenue  $p$  and the expected surplus generated by the sale is  $\mathcal{S}(p)/d(p)$ . For the CES demand, the expected surplus conditioning on a sale  $\mathcal{S}(p)/d(p) = p/(b - 1)$  is *perfectly aligned* with the revenue  $p$  as  $b > 1$ . In this case, when the firm maximizes its total expected revenue using dynamic pricing, it also coincidentally maximizes the total consumers' surplus. For exponential demand, the expected surplus on a sale is  $\mathcal{S}(p)/d(p) = \theta$ , which is *independent* of the revenue. Thus, consumers' surplus is not very sensitive to the pricing policy. It turns out that dynamic pricing still increases consumers' surplus by increasing sales, but not as much as the CES demand. For the linear demand,  $\mathcal{S}(p)/d(p) = (1 - ap)^+/2$  is *negatively aligned* with the firm's revenue  $p$  earned from a sale. When the firm maximizes its revenue using dynamic pricing, it may hurt customers because their objectives are misaligned. Nevertheless, even in the extreme case of  $c = 1$  and a large value of  $T$ , we see that the dynamic pricing policy  $p_v^*$  generates more welfare than the fixed pricing policy  $p_f^*$ , and thus even here, the zero-sum hypothesis fails to hold.

### 4.4. When Is Dynamic Pricing Win-Win?

We now formalize the intuition stated in Section 4.3. As we focus on the revenue-maximizing dynamic pricing policy, the resulting allocation is the rightmost point on the Pareto frontier. If the Pareto frontier is very long, then the two allocations of  $\pi = 0$  and  $\pi = 1$  are far apart on the  $V - S$  plane. In this case, the consumers' surplus generated by  $p_v^*(t, x)$  corresponding to  $\pi = 1$  may be significantly lower than that generated by  $p_s^*(t, x)$ , which is the surplus-maximizing pricing policy and corresponds to  $\pi = 0$ . In particular, the allocation of the revenue-maximizing static policy, which is a point inside the Pareto frontier, can be higher than the allocation of  $p_v^*$ , implying higher consumers' surplus. In fact, no pricing policies can be near optimal for both the firm and customers. On the other hand, if the Pareto frontier is short, then the two allocations corresponding to  $\pi = 0$  and  $\pi = 1$  are close. As a result, the revenue-maximizing dynamic pricing policy generates a level of consumers' surplus that is only slightly lower than the maximal surplus. It is unlikely that the allocation of  $p_f^*$  is placed higher than  $p_v^*$  on the plane. In the extreme case, the frontier collapses to a single point for the CES demand function. The revenue-maximizing

pricing policy also maximizes consumers' surplus or any  $W_\pi(T, c)$ .

To find demand functions whose Pareto-efficient frontier is short, we resort to the intuition discussed in Section 4.3: if the welfare allocation conditioning on a sale  $(p, \mathcal{S}(p)/d(p))$  is aligned, then the term  $\pi p + (1 - \pi)\mathcal{S}(p)/d(p)$  does not change significantly for  $\pi \in [0, 1]$ , and maximizing  $W_\pi$  yields similar allocations.

**Theorem 2.** *If  $pd(p)/(\mathcal{S}(p)) \in [a, b]$  for all prices  $p$  that may be used, then*

$$S(T, c | p_V^*) \geq \frac{a}{b} S(T, c | p_S^*).$$

Theorem 2 implies that the revenue-maximizing pricing policy almost maximizes consumers' surplus if the ratio  $a/b$  is close to one. In this case, it is unlikely that a static price can achieve higher consumers' surplus than the revenue-maximizing pricing policy. In fact, under the condition in Theorem 2, we can show that  $V(T, c | p_S^*) \geq (a/b)V(T, c | p_V^*)$  by the same argument. That is, the surplus-maximizing pricing policy also achieves a fraction  $a/b$  of the maximal revenue. Therefore, the Pareto frontier is contained in a rectangle of length  $(1 - a/b)V(T, c | p_V^*)$  and height  $(1 - a/b)S(T, c | p_S^*)$ . As  $a/b$  approaches 1, the rectangle shrinks and the Pareto frontier becomes shorter. As a result, dynamic pricing is more likely to be win-win. For the linear demand function  $d(p) = 1 - p$ , we have  $pd(p)/\mathcal{S}(p) = 2p/(1 - p)$ . The dynamic price is always greater than  $\hat{p} = 1/2$ . If the price set by the firm ranges from  $1/2$  to  $2/3$ , then  $pd(p)/\mathcal{S}(p) \in (2, 4)$  and the ratio  $a/b = 1/2$ , so the pricing policy generates at least a half of the maximal consumers' surplus.

It is natural to ask what demand functions have a value of  $a/b$  close to one. By Proposition 7,  $a/b = 1$  for the demand function with constant elasticity. Here, demand elasticity  $e(p)$  equals  $-pd'(p)/d(p)$ , which is a standard definition in economics. This implies that the welfare allocation is related to the demand elasticity. Indeed, because  $(pd(p))' = pd'(p) + d(p)$  and  $\mathcal{S}'(p) = -d(p)$ , the demand elasticity reflects the ratio of the derivatives of the revenue function  $pd(p)$  and the consumers' surplus  $\mathcal{S}(p)$ , which is precisely  $e(p) - 1$ . If the elasticity has small variations, so does  $pd(p)/\mathcal{S}(p)$ ; consequently, the welfare allocated to the firm and the consumers  $(p, \mathcal{S}(p)/d(p))$  is well aligned. This intuition is validated in Proposition 9.

**Proposition 9.** *For  $p \geq \hat{p}$ , if  $e(p) \in [e_{\min}, e_{\max}]$ , then*

$$\frac{pd(p)}{\mathcal{S}(p)} \in [e_{\min} - 1, e_{\max} - 1].$$

We restrict our attention to  $p \geq \hat{p}$  because this is the domain of the price that the firm may consider. It also guarantees that  $e(p) \geq 1$  so that the bounds are positive.

## 5. Extensions

In this section, we explore several extensions to our basic case, including network revenue management and a dynamic pricing model for queueing systems.

### 5.1. Network Revenue Management

Gallego and van Ryzin (1997) consider a firm with a set of finite resources that are used to produce a set of products. The firm sets prices dynamically for each product over a finite horizon to maximize its expected revenue. This setting has been studied extensively because of its wide applicability to many industries. In this section, we investigate the welfare implications of the network revenue management problem.

Consider a representative customer who generates utility  $U(\mathbf{q})$  from bundle  $\mathbf{q} \in \mathbb{R}_+^n$ , where the utility function  $U(\cdot): \mathbb{R}_+^n \rightarrow \mathbb{R}_+$  is assumed to be twice differentiable and concave (the Hessian matrix is negative semidefinite). Maximizing her net utility  $\max_{\mathbf{q}} \{U(\mathbf{q}) - \mathbf{p}'\mathbf{q}\}$ , where  $\mathbf{p} \in \mathbb{R}_+^n$  is the price vector of the bundle, gives the demand function  $\mathbf{q} = d(\mathbf{p}) = (\nabla U)^{-1}(\mathbf{p})$ . Consumer's surplus is therefore  $\mathcal{S}(\mathbf{p}) = U(d(\mathbf{p})) - \mathbf{p}'d(\mathbf{p})$ .

Next, we study fluid models to maximize social welfare for the central planner and revenue for the firm. We will use the transformation  $\mathbf{p} = \nabla U(\mathbf{q})$  to cast the problem in terms of  $\mathbf{q}$  instead of  $\mathbf{p}$ . The initial capacity is  $\mathbf{c} \in \mathbb{R}_{\geq 0}^m$  for  $m$  resources. The utilization matrix  $A \in \mathbb{R}_+^{m \times n}$  specifies the resources used by each product—i.e.,  $a_{ij}$  is the amount of resource  $i$  used by one unit of product  $j$ . We again assume a sales horizon of length  $T$ . The fluid welfare and revenue optimization is given by

$$\min_{\mathbf{q}_t \geq 0} \int_0^T [\mathcal{S}(\nabla U(\mathbf{q}_t)) + \mathbf{q}_t' \nabla U(\mathbf{q}_t)] dt = \int_0^T U(\mathbf{q}_t) dt,$$

$$\text{and } \min_{\mathbf{q}_t \geq 0} \int_0^T \mathbf{q}_t' \nabla U(\mathbf{q}_t) dt$$

subject to the same constraint  $\int_0^T A \mathbf{q}_t dt \leq \mathbf{c}$ . We introduce the shadow costs of the products in both problems as the dual variable of the constraint. The following proposition characterizes and compares the optimal solutions to both problems.

**Proposition 10.** *If the number of resources is equal to the number of products, the utilization matrix is invertible, and the shadow costs are positive under optimality in both problems, then the welfare-maximizing and revenue-maximizing pricing policies are identical and time invariant.*

The positivity of the shadow costs implies the scarcity of the resources. In this case, we can show (see the proof of Proposition 10) that the optimal demand rate always clears the market in both problems—i.e., the constraint is binding. However, unlike Section 2, the market-clearing demand rates in the two optimizations are different in general, unless  $m = n$  and  $A$  is an invertible matrix.

## 5.2. A Queuing Model

Our original model in Section 2 is motivated by a system with nonreplenishable capacity. The intuition and insight carry over to fluid queueing systems, which are motivated by the pricing of toll roads and bridges. Central planners or private companies operating roads and bridges charge a price for the entrance of vehicles. The former may be interested in maximizing social welfare, while the latter focuses on the returns of the investment. While the model and the analysis are presented in the online appendix, our findings imply that when the capacity is constrained, the revenue-maximizing toll policy is equivalent to the welfare-maximizing policy. In addition, if the hazard rate of the demand function is decreasing, then the public interest is also best served under this policy. This conclusion provides valuable insights into the ongoing controversies over privatization of toll roads in the United States given the growing concern that road privatization could hurt the public interest (Baxandall et al. 2009).

## 6. Conclusion and Future Research

This paper analyzes the impact of dynamic pricing on social welfare and consumers' surplus. We find that the market-clearing price asymptotically maximizes both social welfare and firm's revenue, as well as consumers' surplus under certain conditions, when demand and capacity are scaled in proportion and capacity is relatively scarce. Moreover, the revenue-maximizing dynamic pricing policy asymptotically maximizes social welfare in the same regime. This is because the Pareto frontier shrinks relative to the scale of the system. As a result, the objectives of the firm and the policy makers becomes aligned. Even without the explicit goal of improving social welfare, the revenue-maximizing dynamic pricing policy benefits consumers compared to the static pricing policy in most cases. We find that this phenomenon depends on how welfare is allocated between the firm and the customer conditioning on a sale and, ultimately, the variation of the demand elasticity.

We briefly explore other settings including network revenue management and pricing in queueing models. However, the following related questions remain to be answered in future research.

- **Strategic customers.** When customers are surplus maximizers, they may wait for lower prices, and the firm must incorporate this strategic behavior to maximize the revenue at equilibrium. Su (2007) shows that the revenue-maximizing pricing policy is socially efficient in the presence of strategic customers. This may be due to the deterministic framework adopted in that paper, as all stocked products are guaranteed to be sold. Cachon and Swinney (2009) argue that the quick response of retailers to account for strategic customers leads to more consumers purchasing at higher prices. The impact on social welfare is therefore ambiguous.

On the other hand, Chen and Farias (2018) and Chen et al. (2018) study firms' revenue-maximization problems in the presence of customer strategic behaviors. They establish that a simple fixed pricing policy can deter strategic behavior and is optimal when the market size is large. It somehow justifies the use of fixed pricing, not from the consumers' but from the firm's point of view. In general, surplus-maximizing strategic consumers and revenue-maximizing firms do not necessarily lead to more social welfare at equilibrium compared to the myopic case, as demonstrated by the prisoner's dilemma. Comparative statics are needed to examine the welfare function as the level of customers' rationality varies. The dynamic price and static price when consumers are strategic can also be compared. A static price induces consumers to behave myopically and may eliminate inefficiency because of strategic waiting and discounting at the cost of lower revenues for the firm.

- **Competition.** The literature on revenue management (see, e.g., Gallego and Hu 2014, Levin et al. 2009) has been focused on oligopolistic competition and Nash equilibria. However, welfare implications have been scarcely discussed. We expect firms to earn lower profits and provide higher consumer surplus in a competitive setting. It is therefore plausible that social welfare increases under competition. In addition, if capacity is relatively scarce, we would expect the pricing policies at a Nash equilibrium to be very similar to that resulting from a central planner managing the aggregate capacity. On the other hand, it is less clear whether a dynamic price competition results in increased social welfare compared to a static price competition in which firms commit to a fixed price over the horizon. Possible reasons for firms using static prices include government regulations and agreements between firms. Unlike the monopolistic setting, static prices do not necessarily generate less revenues than dynamic prices if other competitors commit to static prices as well. We therefore believe that social welfare may either increase or decrease when all firms switch from static prices to dynamic prices.

- **Other forms of dynamic pricing or price discrimination.** For example, intertemporal price discrimination is another scheme commonly used in the airline industry. As the demand function varies over time, a firm adjusts prices according to not only the remaining capacity but also the customers' time-varying WtP. As pointed out in the literature review, the welfare effects of intertemporal price discrimination remain unclear.

- **Because customers have different arrival time and WtP,** the firm or the central planner may be concerned about who benefit from dynamic pricing. According to Pang et al. (2015), the revenue-maximizing price is a submartingale before the stopping time when the inventory drops to 1, and then becomes a supermartingale. However, it is still unclear whether a customer

arriving early enjoys higher surplus than a latecomer. To shed light on this question, we conduct a numerical example whose details are provided in the online appendix. The numerical example suggests that an early customer tends to enjoy higher surplus. The same question is also relevant when there are strategic customers and/or the demand function is time varying.

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### Endnote

<sup>1</sup>For example,  $d(p)$  is upper semicontinuous and  $d(p) = o(1/p)$  as  $p \rightarrow \infty$ , which is assumed to hold throughout the paper.

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