An introduction to the Black-Scholes model

1 Setting

We assume that the market contains two assets.

- A risk-free asset \( P_t \) (typically a bond), which gives an interest rate \( r \). Mathematically, it means that \( P_t \) verifies the ODE
  \[
  dP_t = rP_t \, dt
  \]
  so that
  \[
  P_t = P_0e^{rt}.
  \]

- A risky asset \( S_t \) (typically a stock), with drift \( \mu \) and volatility \( \sigma \). This means that \( S_t \) verifies the SDE
  \[
  dS_t = S_t(\mu \, dt + \sigma \, dB_t)
  \]
  As we have seen, using Ito’s formula, it is easy to check that a solution (actually the solution started from \( S_0 \)) to this equation is
  \[
  S_t = S_0 \exp \left( \sigma B_t + \left( \mu - \frac{\sigma^2}{2} \right) t \right).
  \]

We leave aside all the financial assumptions concerning the cost of buying these assets, their availability, and so on. We are also allowed to buy negative amounts, that is, borrow money at interest rate \( r \), or “short selling” the stock. However, an important assumption is the absence of arbitrage. Informally, we assume that we cannot make a positive amount of money with no risk. Typically, this occurs if you can buy an asset at some price, and sell it right away at a higher price. Hence, there is a right price for a financial product. Our goal will be to find the right price for the European call, which we now define.

2 Options

A common type of financial products are the so-called options. We will focus specifically on European calls. It is a product that allows you to buy the stock at some later time \( T \), and at a price \( K \). Hence, you bet that, at time \( T \), the stock will be worth more than \( K \). If this is the case, you can then buy at price \( K \) and sell it right away to make a profit \( S_T - K \). If the stock is at a price lower than \( K \), you do not use (called exercise) your option. Hence, we say that the price \( \Phi_T \) of the option at time \( T \) is

- \( S_T - K \) if \( S_T > K \);
- \( 0 \) if \( S_T \leq K \),

which we write

\[
\Phi_T = (S_T - K)^+.
\]

More generally, the price \( \Phi_t \) of the option at time \( t \) is the (minimum) price at which you should sell it. Alternatively, it is the (maximum) price at which you would buy it. The question that we wish to answer now is: what is the price of the option at time \( t < T \)?
3 Portfolio

Now, let us study what strategies we can adopt when buying bonds and stocks. At time $t$, we assume that we have a quantity $\alpha_t$ of stock, and $\beta_t$ of bonds. Of course, we assume that we cannot predict the future, i.e. that $\alpha$ and $\beta$ are $(\mathcal{F}_t)$-adapted, where $\mathcal{F}_t = \sigma(B_s, s \leq t)$. We then have a portfolio $(\alpha, \beta)$. Its value is

$$V_t = V_t(\alpha, \beta) = \alpha_t S_t + \beta_t P_t.$$  \hfill (4)

We will look at a specific type of portfolios, called self-financing. Informally, this means that you can buy stocks and bonds only with the money coming from the portfolio. Mathematically, it means

$$dV_t = \alpha_t dS_t + \beta_t dP_t.$$  \hfill (5)

Assume that there exists a self-financing portfolio $(\alpha, \beta)$, with value $V_t$, which replicates the option, i.e. such that $V_T = \Phi_T$. The point is the following.

**Lemma 3.1.** The price $\Phi_t$ of the option at time $t$ is $V_t$.

**Proof.** By assumption, $V_T = \Phi_T$, i.e. investing $V_t$ at time $t$ in the portfolio allows to make the exact same profit as the option. Hence, if you sell the option at a price $x > V_t$ at time $t$, then no one would buy it: you could invest the smaller amount $V_t$ in the portfolio and make the same profit at the end. On the other hand, you should definitely buy the option if its price is $x < V_t$. You would then make the same profit $\Phi_T$, for a smaller investment. \hfill \qed

Hence, the question boils down to computing $V_t$. We may expect that $V_t$ is a martingale, and this is clearly wrong because of the interest rate. More reasonably, the discounted price $e^{-rt}V_t$ might be, but this is not the case. We may indeed compute the following, using the product formula, (5), (4) and (2):

$$d(e^{-rt}V_t) = -re^{-rt}V_t \, dt + e^{-rt} \, dV_t$$

$$= e^{-rt} (-rV_t \, dt + \alpha_t \, dS_t + \beta_t \, dP_t)$$

$$= e^{-rt} (-r(\alpha_t S_t + \beta_t P_t) \, dt + \alpha_t \, S_t(\mu \, dt + \sigma \, dB_t) + \beta_t \, dP_t)$$

$$= \alpha_t S_t e^{-rt} (-r \, dt + \mu \, dt + \sigma \, dB_t) + e^{-rt} \beta_t (-rP_t + dP_t).$$

The second term vanishes because of (1), and we thus get

$$d(e^{-rt}V_t) = \sigma \alpha_t S_t e^{-rt} \left( \frac{\mu - r}{\sigma} dt + dB_t \right).$$  \hfill (6)

Since there is a $dt$ term, this is not a martingale (except if $\mu = r$, but this has no reason to be true). However, there is a way to turn it into a martingale.

4 Change of measure

Consider a discrete random variable $X$, taking values $x_1, \ldots, x_n \in \mathbb{R}$. What we want to know is its distribution, that is $\mathbb{P}(X = x_i)$ for each $i$. Then, we can for instance compute its expectation

$$\mathbb{E}(X) = \sum_{i=1}^n x_i \mathbb{P}(X = x_i).$$
Now, we can change the way to define the probabilities of events as follows. Consider a nonnegative random variable \( M \) with \( \mathbb{E}(M) = 1 \). We can then introduce a new probability measure \( Q \) by defining, for any event \( A \),
\[
Q(A) = \mathbb{E}(MI_A).
\]
Since \( \mathbb{E}(M) = 1 \), this is indeed a probability measure. We have therefore changed the way we define the probability of events, and thus changed the expectation, variance, and more generally the distribution of our random variables. However, we have not changed the values they take. We write \( \mathbb{E}_Q \) for the expectation under \( Q \). For instance
\[
\mathbb{E}_Q(X) = \sum_{i=1}^{n} x_i Q(X = x_i) = \sum_{i=1}^{n} x_i \mathbb{E}(M 1_{\{X = x_i\}}).
\]
More generally,
\[
\mathbb{E}_Q(X) = \mathbb{E}(MX).
\]
In some sense, we bias the variable \( X \), giving more importance to some events, and less to others. For instance, if \( M > 1 \) when \( X = x \), then \( Q(X = x) \) will be larger than \( P(X = x) \).

**Exercise.** Consider \( X \) such that \( X = -1 \) or \( +1 \) with probability \( 1/2 \). Define \( M = 3/2 \) if \( X = 1 \), and \( M = 1/2 \) if \( X = -1 \), so that \( \mathbb{E}(M) = 1 \). Compute \( Q(X = -1) \), \( Q(X = 1) \) and \( \mathbb{E}_Q(X) \).

**Exercise.** Take \( N \) a standard Gaussian variable and \( M = \exp(\mu N - \mu^2/2) \). For \( \lambda \in \mathbb{R} \), compute \( \mathbb{E}_Q(e^{\lambda N}) \) and deduce, that \( N \) is still Gaussian under \( Q \), with mean \( \mu \) and variance 1. Deduce (in one line) \( \mathbb{E}(MN) \). Check your formula by computing \( \mathbb{E}(MN) \) directly by computing the integral.

## 5 Girsanov theorem

Define, for \( \lambda \in \mathbb{R} \),
\[
M_t = \exp \left( \lambda B_t - \frac{\lambda^2}{2} t \right).
\]
As we know, this is a \((\mathcal{F}_t)\)-martingale, so that \( \mathbb{E}(M_T) = \mathbb{E}(M_0) = 1 \). We can then define a probability measure \( Q \) by \( Q(A) = \mathbb{E}(M_T 1_A) \). Similarly to the second exercise above, Girsanov’s theorem ensures that, under \( Q \), \( B \) is still a Brownian motion, except that a drift has been added.

**Remark.** A little technical note beforehand: as is clear from the definition, any event that has probability 0 under \( P \) still has probability 0 under \( Q \), and vice-versa. The two probabilities are called equivalent. In particular, “almost sure” (a.s) has the same meaning under \( P \) or \( Q \), so do (a.s.) constant r.v., and so on.

**Theorem 5.1. (Girsanov theorem)** Under \( Q \), \( B \) is a Brownian motion with drift \( \lambda \).

We will need the following little result.

**Lemma 5.2.** A process \((Y_t)\) is a \((\mathcal{F}_t)\)-martingale if and only if, for all \( s < t \) and every \( \mathcal{F}_s \)-measurable r.v. \( Z \), we have
\[
\mathbb{E}(Y_t Z) = \mathbb{E}(Y_s Z).
\]
Proof. It is easy to check that, if $Y$ is a $(\mathcal{F}_t)$-martingale, then this holds. The converse is actually the (rigorous) definition of the conditional expectation. \hfill \square

We can then prove the Girsanov theorem. This simple version is sometimes known as the Cameron-Martingale formula.

Proof. Define $W_t = B_t - \lambda t$. This should be a Brownian motion under $Q$. According to Levy’s characterization theorem, it suffices to check that $W$ is a $\mathbb{Q}$-martingale, and that $\langle W \rangle_t = t$. The second part is clear, since $dW_t = dB_t - \lambda dt$, and thus $\langle W \rangle_t = t$ (it is a constant, so it does not change that we look under $\mathbb{P}$ or $\mathbb{Q}$).

As for the first part, note, to begin with, that $dM_t = \lambda M_t dB_t$, $dW_t = dB_t - \lambda dt$, and thus $d\langle W, B \rangle_t = \lambda M_t dt$. Using the product rule, we can then do the following computation (classically, under $\mathbb{P}$):

$$
d(W_t M_t) = W_t dM_t + M_t dW_t + d\langle W, M \rangle_t
= \lambda W_t M_t dB_t + M_t (dB_t - \lambda dt) + \lambda M_t dt
= \lambda W_t M_t dB_t + M_t dB_t.
$$

Hence, $(W_t M_t)$ is a $\mathbb{P}$-martingale.

Using Lemma 5.2, let us now check that $W$ is a $\mathbb{Q}$-martingale (it is definitely not a $\mathbb{P}$-martingale!). We have indeed, for every $c\mathcal{F}_s$-measurable r. v. $Z$, that

$$
\mathbb{E}_\mathbb{Q}(W_t Z) = \mathbb{E}(M_T W_t Z) = \mathbb{E}(\mathbb{E}(M_T W_t Z | \mathcal{F}_t)) = \mathbb{E}(W_t Z \mathbb{E}(M_T | \mathcal{F}_t)) = \mathbb{E}(W_t M_t Z) = \mathbb{E}(W_s M_s Z)
$$

where we use that $M$ is a martingale, and then that $WM$ is also a martingale, and Lemma 5.2. By similar computations, this last term is $\mathbb{E}_\mathbb{Q}(W_s Z)$, and we have the result.

Warning. It is true that $\mathbb{E}_\mathbb{Q}(X) = \mathbb{E}(M_T X)$, but is wrong that $\mathbb{E}_\mathbb{Q}(X | \mathcal{F}_s) = \mathbb{E}_\mathbb{Q}(X M_T | \mathcal{F}_s)$. \hfill \square

6 Conclusion

Recall that we want to compute the price $\Phi_t = V_t$ of our option at time $t$. Let us denote $\lambda = -(\mu - r)/\sigma$ and $W_t = B_t - \lambda t$. We can rewrite (6) as

$$
d(e^{-rt} V_t) = \sigma \alpha_t S_t e^{-rt} dW_t.
$$

Now, define $M_t$ and $Q$ as in the previous section. We have thus that, under $Q$, $B$ is a Brownian motion with drift $\lambda$. Hence, under $Q$, $W_t = B_t - \lambda t$ is a standard Brownian motion. From the equation above, $(e^{-rt} V_t)$ is thus a martingale! In particular

$$
V_t = e^{rt} e^{-rt} V_t = e^{rt} \mathbb{E}_Q(e^{-rT} V_T | \mathcal{F}_t) = e^{r(t-T)} \mathbb{E}_Q((S_T - K)^+ | \mathcal{F}_t).
$$

(7)

Recall from (3) that we have

$$
S_t = S_0 \exp \left(\sigma B_t + \left(\mu - \frac{\sigma^2}{2}\right) t\right)
= S_0 \exp \left(\sigma W_t + \left(\sigma \lambda + \mu - \frac{\sigma^2}{2}\right) t\right)
= S_0 \exp \left(\sigma W_t + \left(r - \frac{\sigma^2}{2}\right) t\right).
$$
We can then plug this into (7) to compute \( V_t \). For simplicity, let us just compute \( V_0 \). The general result can be obtained with the usual technique of writing \( W_T = (W_T - W_t) + W_t \), but this leads to more tedious computations.

**Theorem 6.1.** *The price of the European call at \( t = 0 \) is*

\[
\Phi_0 = S_0 \Psi(d_1) - Ke^{-rT} \Psi(d_2),
\]

where

\[
d_1 = \frac{-\log(K/S_0) + (r + \sigma^2/2)T}{\sigma \sqrt{T}}, \quad d_2 = d_1 - \sigma \sqrt{T} = \frac{-\log(K/S_0) + (r - \sigma^2/2)T}{\sigma \sqrt{T}},
\]

and

\[
\Psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-x^2/2} \, dx
\]

is the cumulative distribution function of a standard normal variable.

**Proof.** For \( t = 0 \), the filtration \( \mathcal{F}_0 \) contains no information, and (7) then gives

\[
V_0 = e^{-rT} E_Q((S_T - K)^+) = e^{-rT} \left( E_Q(S_T 1_{\{S_T \geq K\}}) - KE_Q(1_{\{S_T \geq K\}}) \right).
\]

As we mentioned, \( W \) is a Brownian motion under \( Q \), so that \( W_T \) has the distribution of \( \sqrt{T}N \) for \( N \) a standard Gaussian normal variable, and therefore \( S_T \) has the distribution of

\[
S_0 \exp \left( \sigma \sqrt{T} + \left( r - \frac{\sigma^2}{2} \right) T \right) = S_0 e^{\gamma N + \delta},
\]

for \( \gamma = \sigma \sqrt{T} \) and \( \delta = (r - \sigma^2/2)T \).

We then have

\[
E_Q(1_{\{S_T \geq K\}}) = P(S_T \geq K) = P(S_0 e^{\gamma N + \delta} \geq K) = P(N \geq \frac{\log(K/S_0) - \delta}{\gamma}) = P(N \leq \frac{-\log(K/S_0) + \delta}{\gamma}) = \Psi(d_2).
\]

Similarly, we get

\[
E_Q(S_T 1_{\{S_T \geq K\}}) = S_0 E(e^{\gamma N + \delta} 1_{\{S_0 e^{\gamma N + \delta} \geq K\}}) = S_0 E(e^{\gamma N + \delta} 1_{\{N \geq -d_2\}}) = S_0 E(e^{-\gamma N + \delta} 1_{\{N \leq -d_2\}}) = S_0 e^\delta \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-d_2} e^{-x^2/2} \, dx = S_0 e^\delta \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-d_2+\gamma} e^{-x^2/2} e^{\gamma^2/2} \, dx = S_0 e^{\delta + \gamma^2/2} \Psi(d_2 + \gamma) = S_0 e^{\gamma T} \Psi(d_1),
\]
where the last part is classically obtained by completing the squares in the integral. Putting everything together, we get the result.

**Exercise.** Compute $\mathbb{E}_Q(S_T \mathbb{1}_{\{S_t \geq K\}})$ by using the Girsanov theorem (or the second exercise in Section 4).

**Exercise.** Compute $V_t$ for $0 \leq t \leq T$.

7 Comments

One would notice that we overlooked the fact that this computation relies on the possibility to actually find a self-financing portfolio that replicates the option. The price of such a portfolio can be searched for in the form $V_t = f(t, S_t)$. Using Ito’s formula on (4) and the self-financing condition (5) yields a PDE known as the Black-Scholes PDE. It can be solved to show that such a portfolio strategy indeed exists. For more details on all this, see

- Grimett & Stirzaker, Probability and random processes, Section 13.10;
- Chesney & al., Mathematical Methods for Financial Markets;
- The Wikipedia page on the Black-Scholes model.

There exists a lot of literature on this topic, and references abound. As usual, a research on Google is a good to find out more.