Martingale Structure in Counting Process

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1 Introduction

1.1 What is Counting Process?

A Counting Process $N(t)$ is defined as :

- $N(0) = 0$
- $N$ increase by 1 at an event time.
- Value of $N(t)$ is the number of events occurred up to time $t$.

More precisely, let $T_1, T_2, ...$ denote the distinct positive event times, i.e., $T_i$ denote the time of the $i$ times of the event occurred at. The counting process $N(t)$ is defined as :

$$ N(t) = \sum_{i \geq 1} I(T_i \leq t), t \geq 0 $$

or, we can see the picture:

![Figure 1.1: Counting Process](image)

1.2 Why there are a Martingale?

Our first question is :

**Why there are a martingale structure in the counting process ?**

Since the event time $\{T_i\}_{i \in \mathbb{N}}$ is increasing , it is clear from the definition of counting process that $N(t)$ is an increasing positive integer-valued function. Surprisingly, this simple observation tell us more that

**The counting process actually is a submartingale.**

To reveal this fact, let $\mathcal{F}_t$ denote the information up to time $t$. Since $N(t)$ is increasing, for $s \leq t$, we have

$$ N(t) \geq N(s) $$

Next, since $N(t)$ is positive and is determined by the information up to time $t$, take conditional expectation on $\mathcal{F}_s$, the information up to time $s$, gives us

$$ E[N(t)|\mathcal{F}_s] \geq E[N(s)|\mathcal{F}_s] = N(s) $$

for all $s \leq t$. Therefore we see that the counting process is a submartingale. Thanks to the well-known **Doob-Meyer decomposition theorem**, which tells that

Submartingale = Martingale + Predictable Process.

it means we can decompose the counting process into the martingale part and the predictable part.
2 Martingale Structure in Counting Process

To understanding the martingale structure in counting process more deeply, we need some theory, which introduced in this section, to make the statement precise and rigorous.

2.1 Predictable Process.

Here we gives the definition of predictable process. In words,

Predictable Process = Left continuous Adapted Process

More precisely,

1. Given a filtration $\mathcal{F}_t$, a process $\{X(t)\}$ is called $\{\mathcal{F}_t\}$-adapted if for all $t$,

   $$X(t) \in \mathcal{F}_t$$

2. A process $\{X(t)\}$ is left continuous if for all $t$,

   $$\lim_{s \uparrow t} X(s) = X(t)$$

3. A process $\{X(t)\}$ is $\{\mathcal{F}_t\}$-predictable if it is a left continuous $\{\mathcal{F}_t\}$-adapted process.

2.2 Intensity Process.

Since the counting process "jumps" at the event times, the "jumps" plays a crucial role in counting process. How to describe the "jumps"? We introduce the intensity process $\lambda(t)$ of $N(t)$, which defined as:

$$\lambda(t) = \lim_{\Delta t \downarrow 0} \frac{P(N(T + \Delta t) - N(t) = 1|\mathcal{F}_{t-})}{\Delta t}$$

where $\mathcal{F}_{t-} = \sigma(\cup_{s<t} \mathcal{F}_s)$, the information just before time $t$. The intensity process $\lambda(t)$ describes the instantaneous rate of even-occurring and it is clear that the intensity process is $\{\mathcal{F}_{t-}\}$-adapted.

Also, we define the Integrated Intensity Process as

$$\Lambda(t) = \int_0^t \lambda(s) ds$$

It will plays the predictable part in the decomposition of the counting process.

Now, we are ready to state the main purpose of this report:

the martingale structure in the counting process.

Theorem 1. Given a counting process $N(t)$ with it's integrated intensity process $\Lambda(t)$, then the difference process $M(t)$ defined as

$$M(t) = N(t) - \Lambda(t)$$

is a martingale.

The proof of this main theorem if given in the next subsection.
2.3 Derive Martingale Structure in Counting Process

**Step 1: Find the Increment.** Define

\[ dN(t) = \lim_{\Delta t \to 0} N(T + \Delta t) - N(t) \]

From definition of intensity process

\[ \lambda(t) = \lim_{\Delta t \to 0} \frac{P(N(T + \Delta t) - N(t) = 1|\mathcal{F}_{t-})}{\Delta t} \]

we can write it as

\[ \lambda(t)\Delta t = P(N(T + \Delta t) - N(t) = 1|\mathcal{F}_{t-}) + o(\Delta t) \]

and since \(dN(t)\) is 0-1 process, we obtain:

\[ \lambda(t)dt = P(dN(t) = 1|\mathcal{F}_{t-}) = E[dN(t)|\mathcal{F}_{t-}] \]

which gives

\[ E[dN(t) - \lambda(t)dt|\mathcal{F}_{t-}] = 0 \]

Define

\[ dM(t) = dN(t) - \lambda(t)dt \]

we find the vital equation of this fact

\[ E[dM(t)|\mathcal{F}_{t-}] = 0 \]

**Step 2: Show it is Martingale.** Let \(v < u\) and note \(\mathcal{F}_v \subset \mathcal{F}_{u-}\). We have:

\[ E[dM(u)|\mathcal{F}_v] = E[E[dM(u)|\mathcal{F}_{u-}])|\mathcal{F}_v] = E[0|\mathcal{F}_v] = 0 \]

which means for \(v \leq s \leq t\), we have

\[ E(\Delta_{s,t} M|\mathcal{F}_v) = 0 \]

where \(\Delta_{s,t} M = M(t) - M(s)\). Therefore, for \(s < t\),

\[ E(M(t)|\mathcal{F}_s) = E(M(s) + \Delta_{s,t} M|\mathcal{F}_s) = M(s) \]

Thus, we find

\[ M(t) = N(t) - \int \lambda(t)dt = N(t) - \Lambda(t) \]

is a martingale. i.e.,

The difference between the counting process and the intensity process is a martingale.
3 Finding Martingale

3.1 Example 1: Exponential Jump Process

Problem Setting The first example we consider a counting process of single event

\[ N(t) = I(T \leq t) \]

where the event time \( T \) has law of exponential random variable, with density

\[ f_T(t) = \lambda e^{-\lambda t} \]

Find the Integrated Intensity Process

\[
P(\Delta t) - N(t) = 1|N(t) = 0) = P(t < T \leq t + \Delta t|T > t) = \frac{\int_t^{t+\Delta t} \lambda e^{\lambda u} du}{\int_0^t \lambda e^{\lambda u} du} = 1 - e^{-\lambda \Delta t}
\]

\[
P(\Delta t) - N(t) = 1|N(t) = 1) = P(t < T \leq t + \Delta t|T \leq t) = 0
\]

\[
\lambda(t) = \lim_{\Delta t \to 0} \frac{P(\Delta t) - N(t) = 1|\mathcal{F}_t)}{\Delta t} = (\lim_{\Delta t \to 0} \frac{1 - e^{-\lambda \Delta t}}{\Delta t})I(T \geq t) = \lambda I(T \geq t)
\]

\[
\Lambda(t) = \int_0^t \lambda(u) du = \int_0^t \lambda I(T \geq u) du = \lambda(T \wedge t)
\]

The Martingale

\[
M(t) = N(t) - \Lambda(t) = I(T \leq t) - \lambda(T \wedge t)
\]

3.2 Example 2: One Event Process

Problem Setting The second example we consider a counting process of single event

\[ N(t) = I(T \leq t) \]

where the event time \( T \) has density \( g(t) \) and cdf \( G(t) \), i.e.,

\[ f_T(t) \sim g(t) \text{ and } P(T \leq t) = G(t) \]

Find the Integrated Intensity Process

\[
P(\Delta t) - N(t) = 1|N(t) = 0) = P(t < T \leq t + \Delta t|T > t) = \frac{\int_t^{t+\Delta t} g(u) du}{1 - G(t)}
\]

\[
P(\Delta t) - N(t) = 1|N(t) = 1) = P(t < T \leq t + \Delta t|T \leq t) = 0
\]

\[
\lambda(t) = \lim_{\Delta t \to 0} \frac{P(\Delta t) - N(t) = 1|\mathcal{F}_t)}{\Delta t} = (\lim_{\Delta t \to 0} \frac{\int_t^{t+\Delta t} g(u) du}{1 - G(t)} \frac{1 - G(t)}{\Delta t})I(T \geq t) = \frac{g(t)}{1 - G(t)} I(T \geq t)
\]

\[
\Lambda(t) = \int_0^t \lambda(u) du = \int_0^t \frac{g(u)}{1 - G(u)} I(T \geq u) du = -\log(1 - G(t \wedge T))
\]

The Martingale

\[
M(t) = N(t) - \Lambda(t) = I(T \leq t) + \log(1 - G(t \wedge T))
\]
REFERENCES
