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Levy's Characterization of Brownian Motion

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1 INTRODUCTION

In this introductory section, we would gives the **definition of Brownian Motion** and the **main problem which leads the result 'Levy's Characterization of Brownian Motion'**.

1.1 WHAT IS BROWNIAN MOTION?

Definition 1.1. A stochastic process $B(t,\omega)$ is called a **Brownian Motion** if it satisfies the following conditions:

1. $P(\omega; B(0,\omega) = 0) = 1$.
2. For $0 \le s < t$, the increment
$$B(t) - B(s) \sim \mathcal{N}(0, t-s)$$
3. For $0 \le t_1 < t_2 < \cdots < t_n < \infty$, the increments
$$B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}}$$
are independent.
4. $P(\omega; B(0,\omega)$ is continuous$) = 1$.

In words,

1. Almost all paths of Brownian Motion starts at zero.
2. The increment of Brownian Motion is Gaussian distributed.
3. The incremenst of Brownian Motion are independent.
4. Almost all paths of Brownian Motion are continuous.

1.2 HOW TO DETERMINE WHETHER IT IS A BROWNIAN MOTION OR NOT?

Problem. *Given a stochastic process* $\{M_t\}$ *defined on probability space* $(\Omega, \mathscr{F}, \mathbb{P})$ *with filtration* $\{\mathscr{F}_t\}_{t\in\mathbb{R}^+}$*, how can we tell whether it is a Brownian Motion w.r.t.* $\mathbb{P}$ *?*

From the above definition, we know Brownian Motion is a stochastic process which starts at zero, with continuous paths almost surely and has independent Gaussian distributed increments. In the class, we known Brownian Motion has quadratic variation equals the time index. Surprisingly, the Levy's Characterization of Brownian Motion theorem tells us that, in the class of stochastic process that starts at zero, with continuous paths almost surely, the quadratic variation of stochastic process can characterize the Brownian Motion. I would reveal this wonderful result in next section.

2 Levy's Characterization of Brownian Motion

In this section, we state and proof the theorem which is known as

Levy's Characterization of Brownian Motion.

2.1 Levy's Characterization of Brownian Motion

Theorem 1. *If a stochastic process $\{M_t\}$ defined on probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with filtration $\{\mathscr{F}_t\}_{t\in\mathbb{R}^+}$ satisfies the following conditions:*

1. $P(M_0 = 0) = 1$.
2. *M_t is a continuous martingale w.r.t. the filtration $\{\mathscr{F}_t\}_{t\in\mathbb{R}^+}$ under $\mathbb{P}$*
3. *The quadratic variation $< M_t >= t$ almost surely w.r.t. $\mathbb{P}$.*

then

the stochastic process $\{M_t\}$ is a Brownian Motion.

2.2 Proof

Here, we check the definition of the Brownian Motion.

(i) To show almost all paths starts at zero.

1. This is by the first assumption :

$$P(M_0 = 0) = 1.$$

(ii) To show the increment is Gaussian distributed.

1. Apply **Ito's lemma** to the function $f(t,x) = e^{i\lambda x + \frac{1}{2}\lambda^2 t}$ and assumption $d < M_t >= dt$

$$\begin{aligned} df(t,M_t) &= \underbrace{\partial_t f(t,M_t)}_{=\frac{1}{2}\lambda^2 f(t,M_t)} dt + \underbrace{\partial_x f(t,M_t)}_{=i\lambda f(t,M_t)} dM_t + \frac{1}{2}\underbrace{\partial^2_{xx} f(t,M_t)}_{=-\lambda^2 f(t,M_t)} \underbrace{d < M_t >}_{=dt} \\ &= i\lambda f(t,M_t) dM_t \end{aligned}$$

Hence we find

$$f(M_t,t) = e^{i\lambda M_t + \frac{1}{2}\lambda^2 t} \text{ is a martingale.}$$

2. Use the martingale property of $f(M_t,t) = e^{i\lambda M_t + \frac{1}{2}\lambda^2 t}$, for $s < t$,

$$\begin{aligned} &E[e^{i\lambda M_t + \frac{1}{2}\lambda^2 t} | \mathscr{F}_s] = e^{i\lambda M_s + \frac{1}{2}\lambda^2 s} \\ \rightarrow &E[e^{i\lambda (M_t - M_s)} | \mathscr{F}_s] = e^{-\frac{1}{2}\lambda^2 (t-s)} \\ \rightarrow &E[e^{i\lambda (M_t - M_s)}] = E[E[e^{i\lambda (M_t - M_s)} | \mathscr{F}_s]] = E[e^{-\frac{1}{2}\lambda^2 (t-s)}] = e^{-\frac{1}{2}\lambda^2 (t-s)} \end{aligned} \quad (✂)$$

Hence we find

$$M_t - M_s \sim \mathcal{N}(0, t-s)$$

(III) TO SHOW INDEPENDENT INCREMENT.

1. We can rewrite (✂) as, for $s < t$,

$$E[e^{i\lambda M_t}|\mathscr{F}_s] = e^{i\lambda M_s} e^{-\frac{1}{2}\lambda^2(t-s)}$$

   In particular, for $t_1 < t_2$,

$$E[e^{i\lambda M_{t_2}}|\mathscr{F}_{t_1}] = e^{i\lambda M_{t_1}} e^{-\frac{1}{2}\lambda^2(t_2-t_1)} \tag{✡}$$

2. Let $0 \le t_1 < t_2 < \cdots < t_n < \infty$. Note that

$$\begin{aligned} &i\lambda_1 M_{t_1} + i\lambda_2(M_{t_2} - M_{t_1}) + \cdot + i\lambda_n(M_{t_n} - M_{t_{n-1}}) \\ =&i(\lambda_1 - \lambda_2)M_{t_1} + i(\lambda_2 - \lambda_3)M_{t_2} + \cdot + i(\lambda_{n-1} - \lambda_n)M_{t_{n-1}} + i\lambda_n M_{t_n} \end{aligned} \tag{✳}$$

3. By using (✡) and (✳), we can find

$$\begin{aligned} &E[e^{i\lambda_1 M_{t_1} + i\lambda_2(M_{t_2} - M_{t_1}) + \cdot + i\lambda_n(M_{t_n} - M_{t_{n-1}})}] \\ =&E[e^{i(\lambda_1-\lambda_2)M_{t_1} + i(\lambda_2-\lambda_3)M_{t_2} + \cdots + i(\lambda_{n-1}-\lambda_n)M_{t_{n-1}} + i\lambda_n M_{t_n}}] \\ =&E[E[e^{i(\lambda_1-\lambda_2)M_{t_1} + i(\lambda_2-\lambda_3)M_{t_2} + \cdots + i(\lambda_{n-1}-\lambda_n)M_{t_{n-1}} + i\lambda_n M_{t_n}}|\mathscr{F}_{t_{n-1}}]] \\ =&E[e^{i(\lambda_1-\lambda_2)M_{t_1} + i(\lambda_2-\lambda_3)M_{t_2} + \cdots + i(\lambda_{n-1}-\lambda_n)} E[e^{i\lambda_n M_{t_n}}|\mathscr{F}_{t_{n-1}}]] \\ =&E[e^{i(\lambda_1-\lambda_2)M_{t_1} + i(\lambda_2-\lambda_3)M_{t_2} + \cdots + i(\lambda_{n-1}-\lambda_n)M_{t_{n-1}}} e^{i\lambda_n M_{t_{n-1}}} e^{-\frac{1}{2}\lambda_n^2(t_n - t_{n-1})}] \\ =&E[e^{i(\lambda_1-\lambda_2)M_{t_1} + i(\lambda_2-\lambda_3)M_{t_2} + \cdots + i\lambda_{n-1}M_{t_{n-1}}}] e^{-\frac{1}{2}\lambda_n^2(t_n - t_{n-1})} \\ =&\cdots \\ =&e^{-\frac{1}{2}\lambda_1^2 t_1} e^{-\frac{1}{2}\lambda_2^2(t_2-t_1)} \cdots e^{-\frac{1}{2}\lambda_n^2(t_n - t_{n-1})} \end{aligned}$$

   Hence we find

$$M_{t_1}, M_{t_2} - M_{t_1}, \cdots, M_{t_n} - M_{t_{n-1}} \text{ are independent.}$$

(IV) TO SHOW ALMOST ALL PATHS OF BROWNIAN MOTION ARE CONTINUOUS.

1. This is by the second assumption :

$$M_t \text{ is a continuous martingale w.r.t. the filtration } \{\mathscr{F}_t\}_{t\in\mathbb{R}^+} \text{ under } \mathbb{P}.$$

Finally, we have shown that the stochastic process $\{M_t\}$ is a Brownian Motion.

From the above proof, we can see the key of showing the increment is Gaussian distributed rely on Ito's lemma. In more general framework of stochastic integration theory, the Ito's lemma admits its version for local martingale. Start from this viewpoint, in next section, we explore the analogue of the key of showing the increment is Gaussian distributed, the **Characterization of multidimensional Gaussian processes with independent increments**.

3 Extend Levy's Characterization to Local Martingale.

3.1 Characterization of multidimensional Gaussian processes with independent increments

In the framework of stochastic integration with local martingale, we first state the Levy's Characterization of Brownian Motion for one-dimension and multi-dimension as following:

Levy's Characterization of Brownian Motion

Theorem 2. *Let X be a local martingale with $X_0 = 0$. Then, the following are equivalent.*

1. *X is standard Brownian motion.*
2. *X is continuous and $X_t^2 - t$ is a local martingale.*
3. *X has quadratic variation $[X]_t = t$.*

Levy's Characterization of d-dimensional Brownian Motion

Theorem 3. *Let $X = (X^1, \cdots, X^d)$ be a d-dimensional local martingale with $X_0 = 0$. Then, the following are equivalent.*

1. *X is a Brownian motion.*
2. *X is continuous and $X_t^i X_t^j - \delta_{ij} t$ is a local martingale for $1 \leq i, j \leq d$.*
3. *X has quadratic variation $[X^i, X^j]_t = \delta_{ij} t$ for $1 \leq i, j \leq d$.*

Now, the main result of this section is given as:

Characterization of multidimensional Gaussian processes with independent increments

Theorem 4. *Let $X = (X^1, \cdots, X^d)$ be a d-dimensional local martingale with $X_0 = 0$. Let $\{\Sigma_t\}_{t \geq 0}$ be symmetric $d \times d$ real matrices such that $\Sigma_0 = 0$ and $t \mapsto a^\top \Sigma a$ is continuous and increasing for all $a \in \mathbb{R}^d$. Then, the following are equivalent.*

1. *$X_t - X_s$ is independent of $\mathscr{F}_s$ and normally distributed with mean zero and covariance matrix $\Sigma_t - \Sigma_s$, for all $t > s \geq 0$.*
2. *$X_t^i X_t^j - \Sigma_t^{ij}$ is a local martingale for $1 \leq i, j \leq d$.*
3. *X has quadratic variation $[X^i, X^j]_t = \Sigma_t^{ij}$ for $1 \leq i, j \leq d$.*

3.2 PROOF

PROOF FROM 1. TO 2.

1. Suppose 1., let $s < t$, we have $X_t - X_s \sim \mathcal{N}_p(0, \Sigma_t - \Sigma_s)$ and hence
$$E[(X_t - X_s)(X_t - X_s)^\top] = \Sigma_t - \Sigma_s$$
2. Let $Y = a^T X$ and $Z = b^T X$, we have $(Y_t - Y_s) = a^T(X_t - X_s)$ and $(Z_t - Y_s) = b^T(X_t - X_s)$ and hence
$$E[(Y_t - Y_s)(Z_t - Z_s)^\top] = a^T(\Sigma_t - \Sigma_s)b$$
3. Also note that since $X_t - X_s$ is independent of $\mathscr{F}_s$, we have
$$E[Y_s(Z_t - Z_s) + Z_s(Y_t - Y_s)|\mathscr{F}_s] = Y_s E[Z_t - Z_s|\mathscr{F}_s] + Z_s E[Y_t - Y_s|\mathscr{F}_s] = 0$$
4. Combine above argument, we find
$$E[Y_t Z_t - Y_s Z_s|\mathscr{F}_s] = E[Y_s(Z_t - Z_s) + Z_s(Y_t - Y_s)|\mathscr{F}_s] + E[(Y_t - Y_s)(Z_t - Z_s)^\top] = a^T(\Sigma_t - \Sigma_s)b$$
which can be rewrite as
$$E[Y_t Z_t - a^T \Sigma_t b|\mathscr{F}_s] = Y_s Z_s - a^T \Sigma_s b$$
we find
$$M_t = Y_t Z_t - a^T \Sigma_t b \text{ is a martingale, hence a local martingale.}$$
5. Finally, take $a = e_i$ and $b = e_j$, we find
$$X_t^i X_t^j - \Sigma_t^{ij} \text{ is a local martingale.}$$

PROOF FROM 2. TO 3.

1. Suppose 2., in particular, we have
$$Y_t^2 - a^T \Sigma_t a \text{ is a local martingale.}$$
Also, we know
$$Y_t^2 - [Y]_t \text{ is a local martingale.}$$
Therefore, the difference
$$V_t = [Y]_t - a^T \Sigma_t a \text{ is also a local martingale.}$$
2. Since V_t is difference of continuous increasing process, V_t is a finite variation (FV) process.
3. combine above, by the fact that any continuous FV local martingale is constant, we find
$$[Y]_t = a^T \Sigma_t a$$
4. Using polarization identity, we find
$$\begin{aligned}[a^T X, b^T X]_t &= \frac{1}{4}([(a+b)^T X]_t - [(a-b)^T X]_t) \\ &= \frac{1}{4}((a+b)^T \Sigma_t (a+b) - (a-b)^T \Sigma_t (a-b)) = a^T \Sigma_t b\end{aligned}$$
5. Finally, take $a = e_i$ and $b = e_j$, we find
$$[X^i, X^j]_t = [e_i^T X, e_j^T X]_t = e_i^T \Sigma_t e_j = \Sigma_t^{ij}$$

PROOF FROM 3. TO 1.

1. Suppose 3., set $Y = a^\top X$ we have

$$[Y]_t = a^T \Sigma_t a$$

2. Let $f(x, y) = e^{ix+\frac{1}{2}y}$ and consider the process $M_t = f(Y_t, [Y]_t)$, then

$$M_t = e^{iY_t+\frac{1}{2}[Y]_t} = e^{ia^\top X_t+\frac{1}{2}a^T\Sigma_t a}$$

3. Using Ito's lemma, we have

$$\begin{aligned} dM_t &= f_x(Y_t, [Y]_t)dY_t + f_y(Y_t, [Y]_t)d[Y]_t + \frac{1}{2}f_{xx}(Y_t, [Y]_t)d[Y]_t \\ &= if(Y_t, [Y]_t)dY_t + \frac{1}{2}f(Y_t, [Y]_t)d[Y]_t + \frac{1}{2}(-f(Y_t, [Y]_t))d[Y]_t = iM_t dY_t \end{aligned}$$

   Hence M is a local martingale.

4. Finally, by direct computation

$$\begin{aligned} E[e^{ia^\top(X_t-X_s)}|\mathscr{F}_s] &= E[e^{-ia^\top X_s-\frac{1}{2}a^\top\Sigma_t a+ia^\top X_t+\frac{1}{2}a^\top\Sigma_t a}|\mathscr{F}_s] \\ &= E[M_t e^{-ia^\top X_s-\frac{1}{2}a^\top\Sigma_t a}|\mathscr{F}_s] \\ &= M_s e^{-ia^\top X_s-\frac{1}{2}a^\top\Sigma_t a} \\ &= e^{-ia^\top X_s+\frac{1}{2}a^\top\Sigma_s a} e^{-ia^\top X_s-\frac{1}{2}a^\top\Sigma_t a} \\ &= e^{\frac{1}{2}a^\top(\Sigma_s-\Sigma_t)a} \end{aligned}$$

   we find

$$X_t - X_s \sim \mathcal{N}_p(0, \Sigma_t - \Sigma_s)$$

References

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