APM 384: Problem Sheet 1

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On this first sheet no exercises are assessed and all should be fairly simple, except possibly the last one. You are encouraged to solve all problems and discuss your difficulties in the tutorials.

1. Check that the function $u(t) = u_0(t)e^{\kappa t}$ solves the system
$$\begin{aligned} u'(t) &= \kappa u(t) \\ u(0) &= u_0. \end{aligned}$$

2. Prove Proposition 1.1 from lectures, that is show that if u_1 and u_2 are two solutions to the same linear homogeneous differential equation, c_1, c_2 are constants and $u = c_1u_1 + c_2u_2$ then u also solves the differential equation.

3. For each of the following PDEs, state its order and whether it is linear, or even linear homogeneous:
   (a) $\frac{\partial u}{\partial t} + \frac{\partial u}{\partial t} = 0$
   (b) $\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial y}\right)^2 = 3$
   (c) $\frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} = x^2 - t^2$.

4. Let u be a function of three variables x, y, z. Check that the directional derivatives in directions $\vec{i}, \vec{j}, \vec{k}$ are $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$ respectively.

5. Let $u(x, y, z) = x^2y^3e^{x-y}\cos(z)$.
   (a) Compute ∇u and Δu
   (b) Hence compute the directional derivative of u in direction $\vec{a}$, i.e. $\vec{a} \cdot \nabla u$, where $\vec{a} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}, e_1$ respectively..

6. Show that if the curve $\gamma(t) = \big(x(t), y(t)\big)$ has $\begin{pmatrix} a(x, y) \\ b(x, y) \end{pmatrix}$ as its tangent vector field, i.e.
$$\frac{d}{dt}\gamma(t) = \begin{pmatrix} a(x(t), y(t)) \\ b(x(t), y(t)) \end{pmatrix}$$

then $\frac{d}{dt}u(\gamma(t) = \begin{pmatrix} a(x,y) \\ b(x,y) \end{pmatrix} \cdot \nabla u(x,y)$.

We did not get to the end of our discussion of the method of characteristics Hence you may wish to defer attempting the following exercises until after the lecture on Monday.

7. (a) Find the general solution to the PDE

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \tag{1}$$

   (b) Hence solve (1) subject to the boundary condition $u(0,y) = \cos(y)$.

8. (a) Find the general solution to the PDE

$$\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = u. \tag{2}$$

   (b) What solution(s) remain when you apply the boundary condition $u(0,y) = y$?

9. (harder) Consider the following PDE for a function u of three arguments:

$$\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$$

   subject to the boundary condition $u(0,y,z) = y+z$. Extend the method of characteristics we discussed in class to the three-dimensional setting and solve this system.