Problem Sheet 1: Solutions

APM 384

Autumn 2014

Throughout this problem sheet let V_1, V_2, V_3 be vector spaces.

1. *Let $T_1, T_2 \colon V_1 \longrightarrow V_2$ be linear operators and $\lambda_1, \lambda_2 \in \mathbb{R}$. Show that the following are also linear operators:*

   (a) $\lambda_1 T_1$

   (b) $T_1 + T_2$

   (c) $\lambda_1 T_1 + \lambda_2 T_2$.

   For each part we need to show that the resulting operator S satisfies $S(\mu_1 v_1 + \mu_2 v_2)$ for all $\mu_1, \mu_2 \in \mathbb{R}$ and $v_1, v_2 \in V_1$. But this is just definition chasing. For example, letting $S = \lambda_1 T_1$:

$$\begin{aligned} S\left(\mu_1 v_1 + \mu_2 v_2\right) &= \lambda_1 T_1\left(\mu_1 v_1 + \mu_2 v_2\right) = \lambda_1\left[\mu_1 T_1\left(v_1\right) + \mu_2 T_1\left(v_2\right)\right] \\ &= \lambda_1 \mu_1 T_1\left(v_1\right) + \lambda_1 \mu_2 T_1\left(v_2\right) = \mu_1 S\left(v_1\right) + \mu_2 S\left(v_2\right). \end{aligned}$$

   The other parts are solved in a similar manner.

2. *Let $T_1 \colon V_1 \longrightarrow V_2$ and $T_2 \colon V_2 \longrightarrow V_3$ be linear operators. Show that their composition $T_2 \circ T_1 \colon V_1 \longrightarrow V_3$ is also a linear operator.*

   This is once more just definition chasing: let $\mu_1, \mu_2 \in \mathbb{R}$ and $v_1, v_2 \in V_1$. Then, by linearity of T_1 and T_2,

$$\begin{aligned} \left(T_2 \circ T_1\right)\left(\mu_1 v_1 + \mu_2 v_2\right) &= T_2\left(T_1\left(\mu_1 v_1 + \mu_2 v_2\right)\right) = T_2\left(\mu_1 T\left(v_1\right) + \mu_2 T_1\left(v_2\right)\right) \\ &= \mu_1 T_2\left(T_1\left(v_1\right)\right) + \mu_2 T_2\left(T_1\left(v_2\right)\right) \\ &= \mu_1\left(T_2 \circ T_1\right)\left(v_1\right) + \mu_2\left(T_2 \circ T_1\right)\left(v_2\right) \end{aligned}$$

3. *Show that u, v are solutions to the same linear homogeneous differential equation then so is $\lambda u + \mu v$ for any $\lambda, \mu \in \mathbb{R}$.*

   By definition this means that there exists some linear differential operator $T \colon \mathcal{C}^\infty\left(\mathbb{R}^d\right) \longrightarrow \mathcal{C}^\infty\left(\mathbb{R}^d\right)$ such that $T(u) = 0$ and $T(v) = 0$. But, by linearity of T it follows that

$$T\left(\lambda u + \mu v\right) = \lambda T(u) + \mu T(v) = \lambda \cdot 0 + \mu \cdot 0 = 0$$

   i.e. $\lambda u + \mu v$ also solves the differential equation.

4. *Show that for any* $j \in \{1, \ldots, d\}$

$$e_j \nabla f(\vec{x}) = \frac{\partial f}{\partial x_j}(\vec{x})$$

For any vector $\vec{v} \in \mathbb{R}^d$ we have $e_j \cdot \vec{v} = v_j$ (the j^{th} coordinate of v). But the j^{th} co-ordinate of ∇f is $\frac{\partial f}{\partial x_j}$, so we are done.

5. *Let* $u \colon \mathbb{R}^3 \longrightarrow \mathbb{R}$ *be defined by* $u(x, y, z) = x^2 y^3 e^{x-y} \cos(z)$.

   (a) *Compute* ∇u *and* Δu.

   (b) *Hence compute the directional derivative of* u *in direction* $\vec{a}$, *where* $\vec{a} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}, e_1$ *respectively.*

   (a) We have

$$\nabla u = \begin{pmatrix} (2+x)xy^3 e^{x-y}\cos(z) \\ (3-y)x^2y^2e^{x-y}\cos(z) \\ -x^2y^3e^{x-y}\sin(z) \end{pmatrix}$$
$$\Delta u = (x^2+4x+2)y^3e^{x-y}\cos(z) + y(6-3y+y^2x^2)e^{x-y}\cos(z) - x^2y^3e^{x-y}\cos(z)$$

   (b) This is a straightforward calculation. For example,

$$\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \cdot \nabla u = (y(2+x) + 2(3-y)x)xy^2e^{x-y}\cos(z) = (2y - xy + 6x)xy^2e^{x-y}\cos(z)$$

6. *Find the solution to the boundary value problem*

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \tag{1}$$

with boundary condition $u(0, y) = \cos(y)$.

We use the method of characteristics. We first need to solve $\dot{X}(t) = 1$ and $\dot{Y}(t) = 1$ subject to $X(0) = 0$ and $Y(0) = y_0$ where y_0 is a parameter to be chosen later. We obtain $X(t) = t$ and $Y(t) = t + y_0$. The next step is to solve $U(t) = 0$ subject to $U(0) = \cos(y_0)$, which immediately leads to $U(t) = \cos(y_0)$ for all t. Let now $(x, y) \in \mathbb{R}^2$ be given. It remains to find values for t and y_0 such that $X(t) = x$ and $Y(t) = y$. This can be achieved by chosing $t = x$ and $y_0 = y - x$. Hence the solution to the BVP given by (1) and the boundary condition is

$$u(x, y) = \cos(y - x)$$

which I leave for you to verify.

7. *Solve the PDE*

$$y\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = u \tag{2}$$

subject to $u(x,0) = x$.

Here we are given a boundary condition as a function of x rather than y, so we interchange the roles of x and y. So we introduce a parameter x_0 to be chosen later and set $y_0 = 0$. We need to solve $\dot{X}(t) = Y(t)$ and $\dot{Y}(t) = 1$ subject to $(X(0), Y(0)) = (x_0, 0)$. Solving first for Y yields $Y(t) = t$, which transforms the ODE for X into $\dot{X}(t) = t$. Together with the initial condition we obtain $X(t) = \frac{t^2}{2} + x_0$. Next we solve $\dot{U}(t) = U(t)$ subject to $U(0) = x_0$, i.e. $U(t) = x_0 e^t$. For given $x, y \in \mathbb{R}$ we need to choose $t = y$ and $x_0 = x - \frac{y^2}{2}$ in order to have $X(t) = x$ and $Y(t) = y$. Thus the solution to (2) with the boundary condition is given by

$$u(x,y) = \left(x - \frac{y^2}{2}\right) e^y$$

8. *(hard) Consider the following PDE for a function* u *of three arguments:*

$$\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0 \tag{3}$$

subject to the boundary condition $u(0,y,z) = y + z$. *Extend the method of characteristics we discussed in class to the three-dimensional setting and solve this system.*

Following the same approach as in the two-dimensional problem we introduce a parameter t and let X, Y, Z be solutions to the coupled ODEs

$$\begin{aligned} \dot{X}(t) &= 1, & X(0) &= 0 \\ \dot{Y}(t) &= Y(t) & Y(0) &= y_0 \\ \dot{Z}(t) &= Z(t) & Z(0) &= z_0 \end{aligned}$$

where now we have two parameters y_0 and z_0 that we need to specify later. Then we have $X(t) = t$ and $Y(t) = y_0 e^t$ and $Z(t) = z_0 e^t$. Applying the chain rule as before we see that if u satisfies (3) then $U(t) := u\left(X(t), Y(t), Z(t)\right)$ must satisfy $U'(t) = 0$ (check it, using the PDE!) and $U(0) = u\left(0, y_0, z_0\right) = y_0 + z_0$. Thus U doesn't actually depend on t and therefore $U(t) = y_0 + z_0$ for all y_0 and z_0. Now let $x, y, z \in \mathbb{R}$. The last step is to find t, y_0 and z_0 such that $(X(t), Y(t), Z(t)) = (x, y, z)$. This can be achieved by choosing $x = t$ and then $y_0 = ye^{-x}$ and $z = ze^{-x}$. The solution to (3) and the boundary condition is therefore

$$u(x,y,z) = y_0 + z_0 = (y+z)e^{-x}.$$