# Problem Sheet 2

APM 384

September 18, 2014

1. *Find the general solution to the PDE*

$$u_x + u_y = y. \tag{1}$$

*subject to the boundary condition* $u(0, y) = e^y$.

So we have $a(x, y) = b(x, y) = 1$, $c(x, y) = 0$, $g(x, y) = y$ and $x_0 = 0$. Thus our characteristic curves are solutions $(X(t), Y(t), U(t))$ to

$$\dot{X}(t) = t, \qquad X(0) = 0 \tag{2}$$

$$\dot{Y}(t) = t, \qquad Y(0) = y_0 \tag{3}$$

$$\dot{U}(t) = Y(t) \qquad U(0) = e^{y_0}. \tag{4}$$

Solving first (2) and (3) yields $X(t) = t$ and $Y(t) = t + y_0$ (Step 1). This transforms (4) to $\dot{U}(t) = t + y_0$ which, together with the initial condition yields $U(t) = \frac{t^2}{2} + y_0 t + e^{y_0}$ (Step 2). Now we solve, for given $x, y \in \mathbb{R}$, the equations $X(t) = x$ and $Y(t) = y - x$, i.e. $t = x$ and $y_0 = y - x$ (Step 3). Putting it all together yields

$$u(x, y) = \frac{x^2}{2} + (y - x)x + e^{y-x}.$$

I leave it up to you to perform Step 4, i.e. check that this function indeed solves (1).

2. *Consider the PDE*

$$Au_x + Bu_y + Cu = G \tag{5}$$

*where* $A, B, C, G$ *are non-zero constants (i.e. don't depend on the arguments* $x$ *and* $y$*). Solve* (5) *together with the boundary condition* $u(x, 0) = \sin(x)$.

In this exercise the boundary condition is given at a particular value of $y$, rather than $x$. So we need to interchange the roles of $x_0$ and $y_0$, i.e. we choose $y_0 = 0$ and leave $x_0$ as a parameter, to be chosen in Step 3. Step 1: We need to solve $\dot{X}(t) = A$ subject to $X(0) = x_0$ and $\dot{Y}(t) = B$ subject to $Y(0) = 0$. Thus we have $X(t) = At + x_0$ and $Y(t) = Bt$. Step 2: We solve $\dot{U}(t) = C - Gt$, i.e. $U(t) = \frac{G}{C} + \kappa e^{-Ct}$ for some constant $\kappa$. The initial condition $U(0) = \sin(x_0)$ implies that $\kappa = \sin(x_0) - \frac{G}{C}$. Step 3: Solving $X(t) = x$ and $Y(t) = y$ for $t$

and $x_0$ yields $t = \frac{y}{B}$ and $x_0 = x - \frac{A}{B}y$. Step 4: putting things together we need to check that

$$u(x,y) = \frac{G}{C} + \left[\sin\left(x - \frac{A}{B}y\right) - \frac{G}{C}\right] e^{-\frac{C}{B}y}$$

satisfies (5) and the boundary condition.

3. *Solve the PDE*

$$u_x + xu_y = x^2. \tag{6}$$

*subject to the boundary condition* $u(0,y) = y$.

Here we have $a(x,y) = 1$, $b(x,y) = x$, $c(x,y) = 0$, $g(x,y) = y$ and $y_0 = 0$. Step 1: Solving $\dot{X}(t) = 1$ subject to $X(0) = 0$ yields $X(t) = t$. Next we solve $\dot{Y}(t) = X(t) = t$ with $Y(0) = y_0$ and obtain $Y(t) = \frac{t^2}{2} + y_0$. Step 2: We have to solve $\dot{U}(t) = X(t)^2 = t^2$ subject to $U(0) = y_0$. I.e. $U(t) = \frac{t^3}{3} + y_0$. Step 3: $t = X(t) = x$ and $\frac{t^2}{2} + y_0 = Y(t) = y$ are obtained by setting $t = x$ and $y_0 = \frac{x^2}{2} - y$. Step 4: check that

$$u(x,y) = \frac{x^3}{3} + \frac{x^2}{2} - y$$

satisfies (6).

4. *Find the general solution to the PDE*

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0. \tag{7}$$

Here we don't have a boundary condition so we impose an artificial one: we note that $u(0,y)$ is a function of $y$ only, which we denote by $f(y)$. Of course, we will not be able to say anything $f$ unless we are also given a boundary condition. Let us now apply the method of characteristics: in Step 1 we solve $\dot{X}(t) = \dot{Y}(t) = t$ subject to $X(0) = 0$ and $Y(0) = y_0$, i.e. $X(t) = t$ and $Y(t) = t + y_0$. Step 2: the ODE $\dot{U}(t) = 0$ with initial condition $U(0) = f(y_0)$ has the constant function $U(t) = f(y_0)$ as its only solution. Step 3: $t = x$ and $y_0 = y - x$. Step 4: So the general solution to (7) is given by $u(x,y) = f(x-y)$, for an arbitrary continuously differentiable function $f$.

5. *Recall that, on the way to deriving the heat equation we arrived at*

$$c\rho\frac{\partial u}{\partial t} = K_0\frac{\partial^2 u}{\partial x^2}. \tag{8}$$

*where* $c, \rho, K_0$ *were positive constants. Suppose that a function* $u(t,x)$ *satisfies* (8) *together with the boundary condition*

$$-K_0\frac{\partial u}{\partial x}(t,0) = -H\left[u(t,0) - u_B(t)\right] \tag{9}$$

*for some function $u_B$ and another constant $H > 0$. Find constants $\gamma_1, \gamma_2$ such that the function $\widetilde{u}$ defined by $\widetilde{u}(t,x) = u(\gamma_1 t, \gamma_2 x)$ satisfies*

$$\frac{\partial \widetilde{u}}{\partial t} = \frac{\partial^2 \widetilde{u}}{\partial x^2} \tag{10}$$

$$\frac{\partial \widetilde{u}}{\partial x}(t,0) = \widetilde{u}(t,0) - \widetilde{u}_B(t) \tag{11}$$

*where (of course!) $\widetilde{u}_B(t) = u_B(\gamma_1 t)$. Make sure you justify your answer, i.e don't just write down $\gamma_1$ and $\gamma_2$ but rather prove that your claim is correct.*

Using the chain rule we see that $\frac{\partial \widetilde{u}}{\partial x}(t,0) = \gamma_1 \frac{H}{K_0} [\widetilde{u}(t,0) - \widetilde{u}_B(t)]$ and

$$\frac{\partial \widetilde{u}}{\partial t} = \frac{\gamma_1}{\gamma_2^2} \frac{K_0}{c\rho} \frac{\partial^2 u}{\partial x^2}.$$

So we need to choose $\gamma_1 = \frac{K_0}{H}$ and $\gamma_2 = \frac{K_0}{\sqrt{c\rho H}}$.

6. *Consider the ODE $\phi''(x) = -\lambda\phi(x)$ and recall that if $\lambda < 0$ the general solution is given by*

$$\phi(x) = A\cosh\left(\sqrt{-\lambda}x\right) + B\sinh\left(\sqrt{-\lambda}x\right). \tag{12}$$

*for any real numbers $A, B$.*

   (a) *Show that if we additionally impose the boundary condition $\phi(0) = \phi(L) = 0$ for some $L > 0$ then we must have $A = B = 0$, i.e. the only solution is the zero function.*

   Since $\cosh(0) = 1$ and $\sinh(0) = 0$, the condition $\phi(0) = 0$ immediately yields $A = 0$. Then $\phi(L) = 0$ is equivalent to $B\sinh\left(\sqrt{-\lambda}L\right) = 0$, but since $L \neq 0$ and the hyperbolic sine only vanishes at the origin we must have $B = 0$

   (b) *Show that the same is true for the boundary condition $\phi'(0) = \phi'(L) = 0$.*
   – This follows similarly.

7. *Let $f\colon [0,L] \longrightarrow \mathbb{R}$ be continuous. Recall that the statement that $f$ is continuous is equivalent to saying that for all $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that for all $x, y \in [0,L]$ with $|x - y| < \delta$ we have $|f(x) - f(y)| < \epsilon$.*

   (a) *Show that if $f(x) > 0$ for some $x > 0$ then there exists $\delta > 0$ such that $[x-\delta, x+\delta] \subseteq [0,L]$ and $f(y) > \frac{f(x)}{2}$ for all $y \in [x-\delta, x+\delta]$.*

   Choose $\epsilon = \frac{f(x)}{2} > 0$ in the definition of continuity. If $\delta > 0$ is large enough so that either $x - \delta < 0$ or $x + \delta > L$ simply choose a smaller value for $\delta$ until neither of these conditions hold.

   (b) *Deduce that $\int_{x-\delta}^{x+\delta} f(y)\,dy > 0$.*

   By monotonicity of the integral $\int_{x-\delta}^{x+\delta} f(y)\,dy > \int_{x-\delta}^{x+\delta} \frac{f(x)}{2} = \delta f(x) > 0$.

(c) *Show similarly that if* $f(x) < 0$ *for some* $x \in [0, L]$ *then there exists* $\delta > 0$ *such that* $[x - \delta, x + \delta] \subseteq [0, L]$ *and* $\int_{x-\delta}^{x+\delta} f(y)\, dy < 0$.

Here we choose $\epsilon = -\frac{f(x)}{2}$ to see that for $|x - y| < \delta$ we have $|f(x) - f(y)| < -\frac{f(x)}{2}$, i.e. $f(y) < \frac{f(x)}{2}$ and then proceed analogously to part (b).

(d) *Deduce that if* $\int_a^b f(y)\, dy = 0$ *for all* $a, b \in [0, L]$ *then* $f(y) = 0$ *for all* $y \in [0, L]$.

We have shown that if $f(x) \neq 0$ for some $x \in [0, L]$ then there must be $a, b \in [0, L]$ such that $\int_a^b f(y)\, dy \neq 0$, which is simply the converse of the claim.