# Problem Sheet 3

APM 384

Autumn 2014

1. *Using the method of separation of variables or otherwise, solve the BVP*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{if } 0 < x < L \text{ and } 0 < y < H \tag{1}$$
$$u(0, y) = u(x, 0) = u(x, H) = 0 \tag{2}$$
$$u(L, y) = g_2(y). \tag{3}$$

2. *Using the method of separation of variables or otherwise, solve the BVP*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{if } 0 < x < L \text{ and } 0 < y < H \tag{4}$$
$$u(0, y) = u(x, 0) = u(L, y) = 0 \tag{5}$$
$$u(x, 0) = f_1(x). \tag{6}$$

These two questions can be solved analogously to the other cases treated in the book and lectures respectively.

3. *Show that* $|z|^2 = z\bar{z}$ *and* $\Re(z) = \frac{1}{2}(z + \bar{z})$ *and* $\Im(z) = \frac{1}{2i}(z - \bar{z})$ *for all* $z \in \mathbb{C}$.
Let us write $z = x + iy$ where $x, y \in \mathbb{R}$. Then

$$z\bar{z} = (x + iy)(x - iy) = x^2 - (iy)^2 = x^2 + y^2 = |z|$$

and further

$$\frac{1}{2}(z + \bar{z}) = \frac{1}{2}(x + iy + x - iy) = \frac{2x}{2} = \Re(z)$$
$$\frac{1}{2i}(z - \bar{z}) = \frac{1}{2i}(x + iy - (x - iy)) = \frac{2iy}{2i} = \Im(z).$$

4. *Let* $D$ *be a domain in* $\mathbb{C}$ *and let* $f\colon D \longrightarrow \mathbb{C}$ *be complex differentiable. For* $x, y \in \mathbb{R}$ *define* $u(x, y) = \Re f(x+iy)$ *and* $v(x, y) = \Im f(x+iy)$. *In this exercise we will show that* $u$ *and* $v$ *are harmonic in* $D$.

   (a) *Using the Cauchy-Riemann equations show that* $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$.

   (b) *Find a similar expression for* $\frac{\partial^2 u}{\partial y^2}$ *and deduce that* $\Delta u(x, y) = 0$ *for all* $(x, y) \in D$.

(c) *Show similarly that $v$ is harmonic in $D$.*

See problem sheet 4.

5. *In this question we obtain Laplace's equation in polar co-ordinates. Suppose that $u(x,y)$ satisfies $\Delta u = 0$ and $v(r,\theta) = u(r\cos(\theta), r\sin(\theta))$. Show that*

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0.$$

   This is a straightforward calculation.

6. (a) *Check that the following three functions are harmonic:*

      i. $u(x,y) = x + 1$
      ii. $u(x,y) = y(x-1)$
      iii. $u(x,y) = xy$.

      This is a straightforward calculation. Alternatively observe that each of these is the real or imaginary part of a complex differentiable function. For example in i. we have $u(x,y) = \Re f(x+iy)$ where $f(z) = z + 1$.

   (b) *For each function $u$ from part (a) check directly that the maximum principle holds, i.e. show that*

$$u(0,0) = \frac{1}{2\pi}\int_0^{2\pi} u(cos(t), sin(t))dt.$$

      Another straightforward computation.

7. *Prove that $e^{i\pi} + 1 = 0$.*

   Take $y = \pi$ in the equation $e^{iy} = \cos(y) + i\sin(y)$.