Problem Sheet 4

APM 384

October 16, 2014

1. *[10 marks] Suppose that* $u\colon (0,\infty)\times\mathbb{R}^3 \longrightarrow \mathbb{R}$ *satisfies the heat equation in three dimensions, i.e.*

$$\frac{\partial u}{\partial t}(t,\vec{x}) = \Delta u(t,\vec{x}). \tag{1}$$

   (a) *Show that for any* $\lambda\in\mathbb{R}$ *the function* $u_\lambda\colon (0,\infty)\times\mathbb{R}^3 \longrightarrow \mathbb{R}$ *defined by* $u_\lambda(t,\vec{x}) = u(\lambda^2 t, \lambda\vec{x})$ *also satisfies* (1).

   This is a straight-forward computation using the Chain Rule.

   (b) *Use this to show that the function* $m\colon (0,\infty)\times\mathbb{R}^3 \longrightarrow \mathbb{R}$ *defined by*

$$m(t,\vec{x}) = \vec{x}\cdot\nabla u(t,\vec{x}) + 2t\frac{\partial u}{\partial t}(t,\vec{x}) \tag{2}$$

   also satisfies (1). Hint: *you may assume that any limit can be interchanged with any derivative that appears in this exercise.*

   We have seen above that u_λ also satisfies the heat equation. Hence, using the hint that we can interchange orders differentiation without justification:

$$\frac{\partial}{\partial t}\left(\frac{\partial u_\lambda}{\partial\lambda}\right) = \frac{\partial}{\partial\lambda}\left(\frac{\partial u_\lambda}{\partial t}\right) = \frac{\partial}{\partial\lambda}(\Delta u_\lambda) = \Delta\left(\frac{\partial u_\lambda}{\partial\lambda}\right).$$

   I.e., $\frac{\partial u_\lambda}{\partial\lambda}$ also satisfies the heat equation for any $\lambda\in\mathbb{R}$. But $m(t,\vec{x}) = \frac{\partial u_\lambda}{\partial\lambda}\big|_{\lambda=1}$, so we are done.

2. [20 marks] For the following question recall that the *interior* of a set $A\subseteq\mathbb{C}$ is defined by

$$A^\circ = \{\eta\in\mathbb{C}; \exists\ r>0\colon B(z,r)\subseteq A\} \tag{3}$$

   and the closure of A by

$$\overline{A} = \{\eta\in\mathbb{C}; \forall\ r>0\, B(z,r)\cap A\neq\varnothing\} \tag{4}$$

   (a) *Show that* $\bar{A} = \mathbb{C}\setminus(\mathbb{C}\setminus A)^\circ$ *for all sets* $A\subseteq\mathbb{C}$.

(b) For $z \in \mathbb{C}$ and $A \subseteq \mathbb{C}$ define

$$d(z, A) = \inf \{|z - \eta| : \eta \in A\}.$$

Show that $d(z, A) = 0$ if and only if $z \in \bar{A}$.

(a) Observe that $z \notin \bar{A}$ if and only if there exists $\epsilon > 0$ such that $B(z, \epsilon) \cap A = \varnothing$ (this is the converse of the definition of the closure). The latter, in turn, is equivalent to the statement $B(z, \epsilon) \subseteq \mathbb{C} \setminus A$ (if a set is disjoint with another, it is contained in the complement of the latter set). Thus $z \notin \bar{A}$ if and only if there exists $\epsilon > 0$ such that $B(z, \epsilon) \subseteq \mathbb{C} \setminus A$, i.e. $z \in (\mathbb{C} \setminus A)^\circ$ (look at the definition of the interior!). In summary we have shown that $\mathbb{C} \setminus \bar{A} = (\mathbb{C} \setminus A)^\circ$, which is equivalent to the claimed statement.

(b) Suppose first that $z \in \bar{A}$. Let $\epsilon > 0$. Then $B(z, \epsilon) \cap A \neq \varnothing$, i.e. there exists $\eta \in A$ such that $|z - \eta| < \epsilon$. Since $d(z, A)$ is the lower bound of $|z - w|$ over all $w \in \mathbb{C}$ it follows that

$$d(z, A) \leq |z - \eta| < \epsilon.$$

Since this is true for all $\epsilon > 0$ it must follow that $d(z, A) = 0$.

Conversely suppose now that $d(z, A) = 0$. Let $\epsilon > 0$. Since $0 < \epsilon$ and 0 is the greatest lower bound of the set $B = \{|z - w| : w \in A\}$ it follows that ϵ cannot be a lower bound for B. I.e. there must exist $\eta \in A$ such that $|z - \eta| < \epsilon$, i.e. $\eta \in B(z, \epsilon)$. We have shown that for all $\epsilon > 0$ the set $A \cap B(z, \epsilon) \neq \varnothing$, i.e. $z \in \bar{A}$.

3. [10 marks]

(a) *Show that $i^{2n} = (-1)^n$ and $i^{2n-1} = (-1)^{n-1} i$ for all $n \in \mathbb{N}$*

This is an easy exercise in proving things by induction. For example, we have $i^0 = 1$ (base case). Suppose now that $i^{2(n-1)} = (-1)^{n-1}$ (induction hypothesis), then we must also have $i^{2n} = i^2 i^{2(n-1)} = -i^{2(n-1)} = (-1)^n$ (inductive step).

(b) *Write down the Taylor series for $\exp(iy)$. By comparing the real and imaginary parts to the Taylor series for $\cos(x)$ and $\sin(x)$ respectively, show that*

$$\exp(iy) = \cos(y) + i \sin(y)$$

for all $y \in \mathbb{R}$.

This is a direct computation: split the Taylor series into the sums over even and odd indices and we obtain:

$$\begin{aligned} e^{iy} &= \sum_{n=0}^{\infty} \frac{(iy)^n}{n!} = \sum_{m=0}^{\infty} \frac{(iy)^{2m}}{(2m)!} + \sum_{\ell=1}^{\infty} \frac{(iy)^{2\ell-1}}{(2\ell-1)!} \\ &= \sum_{m=0}^{\infty} (-1)^m \frac{y^{2m}}{(2m)!} + i \sum_{\ell=1}^{\infty} (-1)^{m-1} \frac{y^{2\ell-1}}{(2\ell-1)!} \\ &= \cos(y) + i \sin(y). \end{aligned}$$

Now, there is a subtlety here involving convergence of the series involved, but as usual we do not concern ourselves with this issue.

(c) *Hence deduce that* $|\exp(iy)| = 1$ *for all* $y \in \mathbb{R}$.

This now follows directly from the fact that $\sin^2(\theta) + \cos^2(\theta) = 1$ for all $\theta \in \mathbb{R}$.

4. [10 marks] *Let* D *be a domain in* $\mathbb{C}$ *and let* $f\colon D \longrightarrow \mathbb{C}$ *be complex differentiable. For* $x, y \in \mathbb{R}$ *define* $u(x,y) = \Re f(x+iy)$ *and* $v(x,y) = \Im f(x+iy)$. *In this exercise we will show that* u *and* v *are harmonic in* D.

(a) *Using the Cauchy-Riemann equations show that* $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$.

We have

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2}{\partial x \partial y}$$

(b) *Find a similar expression for* $\frac{\partial^2 u}{\partial y^2}$ *and deduce that* $\Delta u(x,y) = 0$ *for all* $(x,y) \in D$.

Similarly $\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2}{\partial x \partial y}$, and so $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ as required.

(c) *Show similarly that* v *is harmonic in* D.

This follows analogously.

5. [20 marks] *Let* D *be a domain in* $\mathbb{R}^2$ *and* u *harmonic in* D. *In this exercise we will prove Theorem 4.9 (the mean value property).*

(a) Suppose first that $0 \in D$. *Let* $R = \sup\{r > 0\colon B(0,r) \subseteq D\}$ *and define* $\phi\colon [0,R) \longrightarrow \mathbb{R}$ *by*

$$\phi(r) = \frac{1}{2\pi}\int_0^{2\pi} u(r\cos\theta, r\sin\theta)\, d\theta.$$

What is $\phi(0)$*?*

(b) Prove that for $r \in [0,R)$ *we have*[1]

$$\phi'(r) = \frac{1}{2\pi}\int_0^{2\pi} \begin{pmatrix}\cos(\theta)\\ \sin(\theta)\end{pmatrix} \cdot \nabla u(r\cos\theta, r\sin\theta)\, d\theta.$$

(c) Show that the right hand side equals $\int_{B(0,r)} \Delta u(x,y)\, dxdy$.

(d) Deduce that

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(r\cos\theta, r\sin\theta)\, d\theta$$

for all $r \in [0,r)$.

[1] recall that for $\vec{x},\ \vec{y} \in \mathbb{R}^d$ we define $x \cdot y = \sum_{j=1}^d x_j y_j$

(e) *By shifting variables, show that if* $(x_0, y_0) \in D$ *and* $r > 0$ *are such that* $B((x_0, y_0), r) \subseteq D$ *then*

$$u(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + r\cos\theta, y_0 + r\sin\theta)\, d\theta.$$

(a) Clearly $\phi(0) = u(0)$ (note: here use $u(0)$ as an obvious shorthand for $u(0,0)$, as the latter notation would quickly become tedious!).

(b) This is just the chain rule:

$$\begin{aligned}\phi'(r) &= \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial}{\partial r} u(r\cos\theta, r\sin\theta)\, d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \begin{pmatrix}\cos(\theta)\\ \sin(\theta)\end{pmatrix} \cdot \nabla u(r\cos\theta, r\sin\theta)\, d\theta.\end{aligned}$$

(c) This is Green's theorem.

(d) Since $\Delta u = 0$ it follows that $\phi'(r) = 0$, i.e. ϕ is constant. Since $\phi(0) = u(0)$, the claimed identity follows.

(e) Straightforward.

6. [10 marks] *Let* Ω *be any domain in* $\mathbb{R}^2$. *Let* $v\colon \overline{\Omega} \longrightarrow \mathbb{R}$ *satisfy Laplace's equation in two dimensions with Neumann boundary conditions, i.e.*

$$\Delta v(x, y) = 0 \qquad \text{if } (x, y) \in \Omega \tag{5}$$
$$v(x, y) = f(x, y) \qquad \text{if } (x, y) \in \partial\Omega. \tag{6}$$

Using the maximum principle, prove that v *is the unique solution to* (5) *and* (6).

Suppose that v_1 and v_2 are two solutions to (5) and (6) and set $w = v_1 - v_2$. Our task is to show that we must have $w = 0$. Observe that w is also harmonic inside Ω and moreover $w(x, y) = 0$ for any $(x, y) \in \partial\Omega$. By the maximum principle, the global maximum of w on $\bar{\Omega}$ must lie on $\partial\Omega$, i.e. $w(x, y) \leq 0$ for all $(x, y) \in \bar{\Omega}$. But the same is true for the global minimum, so that also $w(x, y) \geq 0$ for all $(x, y) \in \bar{\Omega}$ and hence $w = 0$.

7. [20 marks] *Consider the 'infinite-string' wave equation*

$$\frac{\partial^2 u}{\partial t^2}(t, x) = c\frac{\partial^2 u}{\partial x^2}(t, x) \qquad \text{for } t > 0 \text{ and } x \in [0, L] \tag{7}$$
$$u(x, 0) = \phi(x) \qquad \text{for } x \in (0, L) \tag{8}$$
$$\frac{\partial u}{\partial t}(x, 0) = \psi(x) \qquad \text{for } x \in (0, L) \tag{9}$$

where $c > 0$ and $\phi, \psi \colon \mathbb{R} \longrightarrow \mathbb{R}$ are given continuously differentiable functions. Show that there can be at most one solution to (7)-(9). Hint: *write a BVP for the difference of two solutions and consider the functional*

$$E[w](t) = \int_{\mathbb{R}} \left[w_t^2(t,x) + c w_x^2(t,x) \right] \, dx. \tag{10}$$

You need to mathematically justify any assumptions that you make. I.e., no marks will be awarded for 'physical reasoning'. (Recall that w_x is just another notation for $\frac{\partial w}{\partial x}$.)

Again we let u_1, u_2 be two solutions and consider $w = u_1 - u_2$. Since the functional $E[w]$ can be viewed as a kind of energy this method is also referred to as the *energy method*. Observe that

$$\frac{\partial}{\partial t} E[w](t) = 2 \int_{\mathbb{R}} \left[\frac{\partial w}{\partial t}(t,x) \frac{\partial^2 w}{\partial t^2}(t,x) + c \, \frac{\partial w}{\partial x}(t,x) \frac{\partial^2 w}{\partial x \partial t}(t,x) \right] \, dx.$$

Integrating the second summand by parts we obtain

$$\frac{\partial}{\partial t} E[w](t) = 2 \int_{\mathbb{R}} \frac{\partial w}{\partial t}(t,x) \left[\frac{\partial^2 w}{\partial t^2}(t,x) - c \, \frac{\partial^2 w}{\partial x^2}(t,x) \right] \, dx = 0.$$

Thus $E[w]$ is actually constant in t, so that $E[w](t) = E[w](0) = 0$. Since the only non-negative continuous function that integrates to zero is the zero function it follows that $\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} = 0$. Together with the initial condition it follows that $w = 0$, as required.