Problem Sheet 5

APM 384

Autumn 2014

On this sheet, all questions are assessed. Exercises 1-2 are worth 5 points, exercises 3-4 are worth 10 points. Solutions are due in the lecture on **Monday 10 November**.

1. *Consider the partial differential equation*

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \alpha u(t,x) + c\rho(x)\frac{\partial u}{\partial t} \tag{1}$$

   (a) *Apply separation of variables, i.e. suppose that $u(t,x) = h(t)\phi(x)$ and find ODEs for $h(t)$ and $\phi(x)$*

   (b) *Show that the ODE for ϕ you have found is a Sturm–Liouville problem*

   (a) Separation of variables yields

$$\frac{h''(t) - ch'(t)}{h(t)} = \frac{\phi''(x)}{\rho(x)\phi(x)} + \alpha = \lambda$$

   which leads to the ODEs

$$h''(t) - ch'(t) = \lambda h(t)$$
$$\phi''(x) + \alpha\phi(x) = \lambda\rho(x)\phi(x).$$

   (b) The latter is in SL form with $p(x) = 1$, $q(x) = \alpha$ and $\sigma = \rho$.

2. *Show that the following ODEs can be written as Sturm–Liouville problems (i.e. find the corresponding functions p, q, σ)*

   (a) $x\phi''(x) + \phi'(x) + \lambda x\phi(x) = 0$

   $x\phi''(x) + \phi'(x) = \frac{\mathrm{d}}{\mathrm{d}x}(x\phi'(x))$, so $p(x) = x$, $q(x) = 0$ and $\sigma(x) = x$. (In particular we need $a > 0$.)

   (b) $(1 - x^2)\phi''(x) - 2x\phi'(x) = (\lambda - x)\phi(x)$

   $p(x) = 1 - x^2$, $q(x) = x$, $\sigma(x) = 1$.

   (c) $\sin(x)\phi''(x) + \cos(x)\phi'(x) + \lambda\phi(x) = 0$.
   $p(x) = \sin(x)$, $q(x) = 0$, $\sigma(x) = 1$

3. *Compute the* Raleigh coefficient

$$\frac{\left[-p\phi_n(x)\phi_n'(x)\right]_a^b + \int_a^b p(x)\left[(\phi_n'(x))^2 - q(x)\phi_n^2(x)\right]\,dx}{\int_a^b \phi_n^2(x)\sigma(x)\,dx} \tag{2}$$

for the special case $p(x) = 1$, $q(x) = 0$ *and* $\sigma(x) = 1$ *and*

(a) $\beta_1 = \beta_3 = 1$, $\beta_2 = \beta_4 = 0$

(b) $\beta_1 = \beta_3 = 0$, $\beta_2 = \beta_4 = 1$.

As mentioned in the lectures, you may take $a = 0$, in which case a direct computation yields $\lambda_n = \left(\frac{n\pi}{b}\right)^2$. Otherwise a simple shift deals with the general a case.

4. *An operator* L *acting on a vector space of functions is said to be* self-adjoint *if*

$$\int_a^b v(x)\,L(u)(x)\,dx = \int_a^b u(x)\,L(v)(x)\,dx \tag{3}$$

for all continuously differentiable functions u *and* v. *State whether the following operators are self-adjoint, acting on the space* $C^3[a,b]$ *of three times continuously differentiable functions* without any boundary condition. *Justify your answers:*

(a) $L(f) = f'' + 3f'''$

(b) $L(f) = f^2$

(c) $L(f) = f$

(a) If we had some homogeneous boundary conditions the operator would be self-adjoint. This way, it is not, because of the boundary terms in the integration by parts formula.

(b) This operator is not self-adjoint: consider, for example, $u(x) = x^2$ and $v(x) = \sin(x)$.

(c) This operator is clearly self-adjoint.