FACULTY OF APPLIED SCIENCE AND ENGINEERING
University of Toronto

APM384F
Partial Differential Equations

Midterm Exam, October 24, 2013

Examiner: J. Ortmann
Duration: 1 hour 45 minutes

NO AIDS EXCEPT RULERS ALLOWED. **Total: 100 marks**

Family name (surname): ______________________

First (Given) name: ______________________

Student number: ______________________

For marker's use only			
Question 1:	/20	Question 4:	/15
Question 2:	/22	Question 5:	/20
Question 3:	/23		
		Total:	/ 100

There are five questions and you should attempt to answer all of them. Write your answers underneath the questions or on the other side of the page. There are some extra pages at the end; if you use these clearly indicate which question you are answering.

You are permitted to use your ruler and other writing equipment. No other aids, electronic or otherwise, are allowed. Each part of a question is labelled with how many marks are available for that part. There are 100 marks in total.

1. (a) [8 marks] Let $V > 0$ be a constant. Find all solutions to the *transport equation*

$$\frac{\partial u}{\partial t} + V\frac{\partial u}{\partial x} = 0. \tag{1}$$

subject to the initial condition $u(0, x) = \cos(x)$.

(b) [12 marks] Use the method of characteristics to find the general solution to the PDE

$$x\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = u(x, y). \tag{2}$$

(a) The method of characteristics yields the solution $u(t, x) = \cos(x - Vt)$

(b) Introducing an artificial boundary condition $u(1, y) = f(y)$ and then applying the method of characteristics we obtain the general solution

$$u(x, y) = xf\left(y - \ln(x)\right)$$

where f is an arbitrary continuously differentiable function.

2. Consider the one dimensional heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad x \in (0, \pi),\ t > 0 \tag{3}$$

subject to the boundary condition

$$u(t, 0) = \frac{\partial u}{\partial x}(t, \pi) = 0 \qquad t > 0 \tag{4}$$

and initial condition $u(0, x) = f(x)$ for all $x \in [0, \pi]$, where f is some continuously differentiable function.

(a) [2 marks] What conditions do f and f' need to satisfy in order for the above to make physical sense?

(b) [5 marks] Assume that the solution u can be written as $u(t, x) = \phi(t)\gamma(x)$. Find a boundary value problem for γ and an ODE for ϕ.

(c) [15 marks] Hence find a solution to PDE, initial and boundary conditions. You may assume that

$$\int_0^\pi \cos\left(\frac{2n+1}{2}x\right)\cos\left(\frac{2m+1}{2}x\right) = \begin{cases} \frac{\pi}{2} & \text{if } m = n \\ 0 & \text{if } n \neq m \end{cases} \tag{5}$$

for all non-negative integers m, n. You do not need to justify any interchange of infinite series with an integration.

(a) We need $f(0) = f'(L) = 0$ for the boundary and initial conditions to match up

(b) Separating variables give $g''(t) = -\lambda g(t)$ with $g(0) = g'(L) = 0$ and $f'(t) = -\lambda f(t)$, where $\lambda \in \mathbb{R}$ is an arbitrary real number.

(c) We first check that $\lambda < 0$ does not yield any solutions: we would obtain solutions of the form $g(t) = A\cosh\left(\sqrt{-\lambda}t\right) + B\cosh\left(\sqrt{-\lambda}t\right)$. Using the boundary conditions leads to $A = B = 0$, which only gives the zero solution (which we will catch with the trigonometric solutions anyway). So we may focus on the case $\lambda \geq 0$. Then we get

$$g(x) = A\cos\left(\sqrt{\lambda}x\right) + B\sin\left(\sqrt{\lambda}x\right).$$

The boundary conditions yield $A = 0$ and then $\lambda_n = \frac{(2n+1)^2\pi^2}{4L^2}$. Thus the solutions for f are of the form $g(t) = Ce^{-\lambda_n t}$. For any sequence $(A_n)_{n\in\mathbb{N}}$ of real numbers, functions of the form

$$u(t, x) = \sum_{n=0}^{\infty} A_n e^{-\frac{(2n+1)^2\pi^2}{4L^2}t}\cos\left(\frac{2n+1}{2}x\right)$$

solve the PDE and boundary conditions. Writing now

$$a_n(f) = \int_0^\pi f(y) \cos\left(\frac{2n+1}{2} y\right) dy$$

and then using the hint shows that a solution to PDE, boundary and initial conditions is given by

$$u(t,x) = \sum_{n=0}^{\infty} a_n(f) e^{-\frac{(2n+1)^2\pi^2}{4L^2} t} \cos\left(\frac{2n+1}{2} x\right).$$

3. Recall that a function $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ is said to be complex differentiable at $z \in \mathbb{C}$ if the limit

$$f'(z) := \lim_{\substack{h\to 0\\ h\in\mathbb{C}}} \frac{f(z+h)-f(z)}{h}$$

exists.

(a) [2 marks] Show that the function $f(z) = (z+1)^2$ is complex differentiable at all $z \in \mathbb{C}$.

(b) [2 marks] Suppose that the functions $u, v\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ are defined by $u(x,y) = \Re f(x+iy)$ and $v(x,y) = \Im f(x+iy)$. Without proof, write down a system of two equations that the partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ and $\frac{\partial u}{\partial y}$ satisfy.

(c) [6 marks] Deduce that u, v must satisfy Laplace's equation $\Delta u(x,y) = \Delta v(x,y) = 0$ for all $x, y \in \mathbb{R}$.

(d) [13 marks] Find a function $g\colon \mathbb{R} \times [1,\infty) \longrightarrow \mathbb{R}$ that satisfies the boundary value problem

$$\begin{aligned} \Delta g(x,y) &= 0, && x \in \mathbb{R},\ y > 1 \\ g(x,1) &= (x+1)^2 - 1 && x \in \mathbb{R} \end{aligned}$$

(a) Observe that

$$\frac{f(z+h)-f(z)}{h} = \frac{(z+1)^2 + 2h(z+1) + h^2 - (z+1)^1}{h} = 2(z+1) + h$$

so that f is complex differentiable everywhere [and $f'(z) = 2(z+1)$].

(b) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

(c) $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right) = \frac{\partial^2 x}{\partial x \partial y} - \frac{\partial^2 x}{\partial x \partial y} = 0$, using the Cauchy–Riemann equations (part (b)). $\Delta v = 0$ is similar.

(d) By part (a) the function $f(z) = (z+1)^2$ is complex differentiable on $\mathbb{C}$, hence by part (c) the function g defined by $g(x,y) = \Re(f(x+iy)) = (x+1)^2 - y^2$ is harmonic on $\mathbb{R}^2$, in particular on $\mathbb{R} \times (1,\infty)$. Moreover $g(x,1) = (x+1)^2 - 1$, so g is the desired function.

4. Recall that the displacement v of a vibrating non-uniform string without external forces and fixed ends at 0 and L satisfies the *non-uniform wave equation*

$$\frac{\partial^2 v}{\partial t^2} = c^2(x)\frac{\partial^2 v}{\partial x^2}, \qquad t > 0,\ x \in (0, L) \tag{6}$$

and boundary condition

$$v(t, 0) = v(t, L) = 0, \qquad t > 0. \tag{7}$$

where $c\colon (0, L) \longrightarrow \mathbb{R}$ is a continuously differentiable function.

(a) [7 marks] Suppose that $v(t, x)$ satisfies (6) and (7) and the initial conditions

$$v(0, x) = f(x), \qquad \frac{\partial v}{\partial t}(0, x) = G(x) \qquad \text{for } x \in (0, L). \tag{8}$$

Suppose that there exist functions $h\colon \mathbb{R} \longrightarrow \mathbb{R}$ and $g\colon [0, \infty) \longrightarrow \mathbb{R}$ such that $v(t, x) = g(t)h(x)$ for all $t, x \in \mathbb{R}$. Derive a second-order ODE for g and a boundary-value problem for h.

(b) [8 marks] Find all solutions to the ODE for g. If your answer depends on a parameter indicate carefully what values the parameter may take.

(a) $g''(t) = -\lambda g(t)$ and $h''(x) + \frac{\lambda}{c(x)} h(x) = 0$ subject to $h(0) = h(L) = 0$.

(b) $g(t) = A\cos\left(\sqrt{\lambda}t\right) + B\sin\left(\sqrt{\lambda}t\right)$ if $\lambda \geq 0$ and $g(t) = A\cosh\left(\sqrt{-\lambda}t\right) + B\sinh\left(\sqrt{-\lambda}t\right)$ if $\lambda < 0$.

5. (a) [9 marks] For each of the following functions $f\colon [-1,1] \longrightarrow \mathbb{R}$ find the coefficients of the Fourier series of f:

   (i) $f(x) = \sin(2\pi x)$

   (ii) $f(x) = x$

   (iii) $f(x) = 1 + x^2$

   (b) [6 marks] Find the Fourier cosine coefficients $A_n(f)$ ($n \in \{0, 1, 2, \ldots\}$) of the function $f\colon [-2,2] \longrightarrow \mathbb{R}$ defined by $f(x) = \sinh(x)$.

   (c) [5 marks] Find the Fourier sine coefficients $B_n(g)$ ($n \in \{1, 2, 3, \ldots\}$) of the function $g\colon [-1,1] \longrightarrow \mathbb{R}$ defined by $g(x) = \frac{1}{4+x^2}$

(a) (i) $B_2(f) = 0$, all other coefficients zero

   (ii) As in class: $A_n(f) = 0$ for all n anb $B_n(f) = \frac{(-1)^{n-1}}{n\pi}$

   (iii) $B_n(f) = 0$ and $A_0(f) = 1$ and $A_n(f) = (-1)^n \frac{4}{n^2\pi^2}$

(b) No computation necessary: sinh is even, so the cosine coefficients are all zero.

(c) No computation necessary: g is odd, so the sine coefficients are all zero.

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