Problem Sheet 2

APM 384

September 21, 2014

On this sheet, questions 1 through 6 are assessed. Please write up your answers, *staple the sheets together* and hand them in to your TA in your tutorial session, either on Thursday **September 25** or Friday **September 26**.

1. Find the general solution to the PDE

$$u_x + u_y = y. \tag{1}$$

subject to the boundary condition $u(0, y) = e^y$.

2. Consider the PDE

$$Au_x + Bu_y + Cu = G \tag{2}$$

where A, B, C, G are non-zero constants (i.e. don't depend on the arguments x and y). Solve (2) together with the boundary condition $u(x, 0) = \sin(x)$.

3. Solve the PDE

$$u_x + xu_y = x^2. \tag{3}$$

subject to the boundary condition $u(0, y) = y$.

4. Find the general solution to the PDE

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0. \tag{4}$$

5. Recall that, on the way to deriving the heat equation we arrived at

$$c\rho \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2}. \tag{5}$$

where c, ρ, K_0 were positive constants. Suppose that a function $u(t, x)$ satisfies (5) together with the boundary condition

$$-K_0 \frac{\partial u}{\partial x}(t, 0) = -H\left[u(t, 0) - u_B(t)\right] \tag{6}$$

for some function u_B and another constant $H > 0$. Find constants γ_1, γ_2 such that the function $\widetilde{u}$ defined by $\widetilde{u}(t,x) = u(\gamma_1 t, \gamma_2 x)$ satisfies

$$\frac{\partial \widetilde{u}}{\partial t} = \frac{\partial^2 \widetilde{u}}{\partial x^2} \tag{7}$$

$$\frac{\partial \widetilde{u}}{\partial x}(t,0) = \widetilde{u}(t,0) - \widetilde{u}_B(t) \tag{8}$$

where (of course!) $\widetilde{u}_B(t) = u_B(\gamma_1 t)$. Make sure you justify your answer, i.e don't just write down γ_1 and γ_2 but rather prove that your claim is correct.

6. Consider the ODE $\phi''(x) = -\lambda\phi(x)$ and recall that if $\lambda < 0$ the general solution is given by

$$\phi(x) = A\cosh\left(\sqrt{-\lambda}x\right) + B\sinh\left(\sqrt{-\lambda}x\right). \tag{9}$$

for any real numbers A, B.

   (a) Show that if we additionally impose the boundary condition $\phi(0) = \phi(L) = 0$ for some $L > 0$ then we must have $A = B = 0$, i.e. the only solution is the zero function.

   (b) Show that the same is true for the boundary condition $\phi'(0) = \phi'(L) = 0$.

7. (not assessed) Let $f\colon [0,L] \longrightarrow \mathbb{R}$ be continuous. Recall that the statement that f is continuous is equivalent to saying that for all $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that for all $x, y \in [0,L]$ with $|x-y| < \delta$ we have $|f(x) - f(y)| < \epsilon$.

   (a) Show that if $f(x) > 0$ for some $x > 0$ then there exists $\delta > 0$ such that $[x-\delta, x+\delta] \subseteq [0,L]$ and $f(y) > \frac{f(x)}{2}$ for all $y \in [x-\delta, x+\delta]$.

   (b) Deduce that $\int_{x-\delta}^{x+\delta} f(y)\,dy > 0$.

   (c) Show similarly that if $f(x) < 0$ for some $x \in [0,L]$ then there exists $\delta > 0$ such that $[x-\delta, x+\delta] \subseteq [0,L]$ and $\int_{x-\delta}^{x+\delta} f(y)\,dy < 0$.

   (d) Deduce that if $\int_a^b f(y)\,dy = 0$ for all $a, b \in [0,L]$ then $f(y) = 0$ for all $y \in [0,L]$.