Problem Sheet 3

APM 384

September 29, 2014

On this sheet, there are no assessed questions. You are encouraged to attempt all questions and ask the TA if you encounter any problems.

1. Using the method of separation of variables or otherwise, solve the BVP

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{if } 0 < x < L \text{ and } 0 < y < H \tag{1}$$
$$u(0, y) = u(x, 0) = u(x, H) = 0 \tag{2}$$
$$u(L, y) = g_2(y). \tag{3}$$

2. Using the method of separation of variables or otherwise, solve the BVP

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{if } 0 < x < L \text{ and } 0 < y < H \tag{4}$$
$$u(0, y) = u(x, 0) = u(L, y) = 0 \tag{5}$$
$$u(x, 0) = f_1(x). \tag{6}$$

3. Show that $|z|^2 = z\bar{z}$ and $\Re(z) = \frac{1}{2}(z + \bar{z})$ and $\Im(z) = \frac{1}{2i}(z - \bar{z})$ for all $z \in \mathbb{C}$.

4. Let D be a domain in $\mathbb{C}$ and let $f\colon D \longrightarrow \mathbb{C}$ be complex differentiable. For $x, y \in \mathbb{R}$ define $u(x, y) = \Re f(x + iy)$ and $v(x, y) = \Im f(x + iy)$. In this exercise we will show that u and v are harmonic in D.

   (a) Using the Cauchy-Riemann equations show that $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$.

   (b) Find a similar expression for $\frac{\partial^2 u}{\partial y^2}$ and deduce that $\Delta u(x, y) = 0$ for all $(x, y) \in D$.

   (c) Show similarly that v is harmonic in D.

5. In this question we obtain Laplace's equation in polar co-ordinates. Suppose that $u(x, y)$ satisfies $\Delta u = 0$ and $v(r, \theta) = u(r\cos(\theta), r\sin(\theta))$. Show that

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0.$$

6. (a) Check that the following three functions are harmonic:

i. $u(x, y) = x + 1$
ii. $u(x, y) = y(x - 1)$
iii. $u(x, y) = xy$.

(b) For each function u from part (a) check directly that the maximum principle holds, i.e. show that

$$u(0,0) = \frac{1}{2\pi} \int_0^{2\pi} u(cos(t), sin(t))dt.$$

7. Prove that $e^{i\pi} + 1 = 0$.