Problem Sheet 4

APM 384

October 7, 2014

On this sheet, all questions are assessed. You may hand in assessed questions in either tutorial (i.e. Thursday or Friday). There are a total of 100 marks available on this sheet, on each question the number of marks is indicated.

1. [10 marks] Suppose that $u\colon (0,\infty)\times\mathbb{R}^3 \longrightarrow \mathbb{R}$ satisfies the heat equation in three dimensions, i.e.

$$\frac{\partial u}{\partial t}(t,\vec{x}) = \Delta u(t,\vec{x}). \tag{1}$$

   (a) Show that for any $\lambda\in\mathbb{R}$ the function $u_\lambda\colon (0,\infty)\times\mathbb{R}^3 \longrightarrow \mathbb{R}$ defined by $u_\lambda(t,\vec{x}) = u(\lambda^2 t, \lambda\vec{x})$ also satisfies (1).

   (b) Use this to show that the function $m\colon (0,\infty)\times\mathbb{R}^3 \longrightarrow \mathbb{R}$ defined by

$$m(t,\vec{x}) = \vec{x}\cdot\nabla u(t,\vec{x}) + 2t\frac{\partial u}{\partial t}(t,\vec{x}) \tag{2}$$

   also satisfies (1).

   Hint: you may assume that any limit can be interchanged with any derivative that appears in this exercise.

2. [20 marks] For the following question recall that the *interior* of a set $A\subseteq\mathbb{C}$ is defined by

$$A^\circ = \{\eta\in\mathbb{C}; \exists\, r>0\colon B(z,r)\subseteq A\} \tag{3}$$

   and the closure of A by

$$\overline{A} = \{\eta\in\mathbb{C}; \forall\, r>0\, B(z,r)\cap A\neq\varnothing\} \tag{4}$$

   (a) Show that $\bar{A} = \mathbb{C}\setminus(\mathbb{C}\setminus A)^\circ$ for all sets $A\subseteq\mathbb{C}$.

   (b) For $z\in\mathbb{C}$ and $A\subseteq\mathbb{C}$ define

$$d(z,A) = \inf\{|z-\eta| : \eta\in A\}.$$

   Show that $d(z,A)=0$ if and only if $z\in\bar{A}$.

3. [10 marks]

(a) Show that $i^{2n} = (-1)^n$ and $i^{2n-1} = (-1)^{n-1}i$ for all $n \in \mathbb{N}$

(b) Write down the Taylor series for $\exp(iy)$. By comparing the real and imaginary parts to the Taylor series for $\cos(x)$ and $\sin(x)$ respectively, show that

$$\exp(iy) = \cos(y) + i\sin(y)$$

for all $y \in \mathbb{R}$.

(c) Hence deduce that $|\exp(iy)| = 1$ for all $y \in \mathbb{R}$.

4. [10 marks] Let D be a domain in $\mathbb{C}$ and let $f \colon D \longrightarrow \mathbb{C}$ be complex differentiable. For $x, y \in \mathbb{R}$ define $u(x,y) = \Re f(x+iy)$ and $v(x,y) = \Im f(x+iy)$. In this exercise we will show that u and v are harmonic in D.

   (a) Using the Cauchy-Riemann equations show that $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$.

   (b) Find a similar expression for $\frac{\partial^2 u}{\partial y^2}$ and deduce that $\Delta u(x,y) = 0$ for all $(x,y) \in D$.

   (c) Show similarly that v is harmonic in D.

5. [20 marks] Let D be a domain in $\mathbb{R}^2$ and u harmonic in D. In this exercise we will prove Theorem 4.9 (the mean value property).

   (a) Suppose first that $0 \in D$. Let $R = \sup\{r > 0 \colon B(0,r) \subseteq D\}$ and define $\phi \colon [0,R) \longrightarrow \mathbb{R}$ by

   $$\phi(r) = \frac{1}{2\pi}\int_0^{2\pi} u(r\cos\theta, r\sin\theta)\,d\theta.$$

   What is $\phi(0)$?

   (b) Prove that for $r \in [0,R)$ we have[1]

   $$\phi'(r) = \frac{1}{2\pi}\int_0^{2\pi} \begin{pmatrix}\cos(\theta)\\ \sin(\theta)\end{pmatrix} \cdot \nabla u(r\cos\theta, r\sin\theta)\,d\theta.$$

   (c) Show that the right hand side equals $\int_{B(0,r)} \Delta u(x,y)\,dxdy$.

   (d) Deduce that

   $$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(r\cos\theta, r\sin\theta)\,d\theta$$

   for all $r \in [0,r)$.

   (e) By shifting variables, show that if $(x_0,y_0) \in D$ and $r > 0$ are such that $B((x_0,y_0), r) \subseteq D$ then

   $$u(x_0,y_0) = \frac{1}{2\pi}\int_0^{2\pi} u(x_0 + r\cos\theta, y_0 + r\sin\theta)\,d\theta.$$

[1] recall that for $\vec{x},\ \vec{y} \in \mathbb{R}^d$ we define $x \cdot y = \sum_{j=1}^{d} x_j y_j$

6. [10 marks] Let Ω be any domain in $\mathbb{R}^2$. Let $v\colon \overline{\Omega} \longrightarrow \mathbb{R}$ satisfy Laplace's equation in two dimensions with Neumann boundary conditions, i.e.

$$\Delta v(x,y) = 0 \qquad \text{if } (x,y) \in \Omega \tag{5}$$

$$v(x,y) = f(x,y) \qquad \text{if } (x,y) \in \partial\Omega. \tag{6}$$

Using the maximum principle, prove that v is the unique solution to (5) and (6).

7. [20 marks] Consider the 'infinite-string' wave equation

$$\frac{\partial^2 u}{\partial t^2}(t,x) = c\frac{\partial^2 u}{\partial x^2}(t,x) \qquad \text{for } t > 0 \text{ and } x \in [0,L] \tag{7}$$

$$u(x,0) = \phi(x) \qquad \text{for } x \in (0,L) \tag{8}$$

$$\frac{\partial u}{\partial t}(x,0) = \psi(x) \qquad \text{for } x \in (0,L) \tag{9}$$

where $c > 0$ and $\phi, \psi\colon \mathbb{R} \longrightarrow \mathbb{R}$ are given continuously differentiable functions. Show that there can be at most one solution to (7)-(9). *Hint:* write a BVP for the difference of two solutions and consider the functional

$$E[w](t) = \int_{\mathbb{R}} \left[w_t^2(t,x) + c w_x^2(t,x)\right]\, dx. \tag{10}$$

You need to mathematically justify any assumptions that you make. I.e., no marks will be awarded for 'physical reasoning'. (Recall that w_x is just another notation for $\frac{\partial w}{\partial x}$.)