Problem Sheet 5

APM 384

Autumn 2014

On this sheet, all questions are assessed. Exercises 1-2 are worth 5 points, exercises 3-4 are worth 10 points. Solutions are due in the lecture on **Monday 10 November**.

1. Consider the partial differential equation

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \alpha u(t,x) + c\rho(x)\frac{\partial u}{\partial t} \tag{1}$$

   (a) Apply separation of variables, i.e. suppose that $u(t,x) = h(t)\phi(x)$ and find ODEs for $h(t)$ and $\phi(x)$

   (b) Show that the ODE for ϕ you have found is a Sturm–Liouville problem

2. Show that the following ODEs can be written as Sturm–Liouville problems (i.e. find the corresponding functions p, q, σ)

   (a) $x\phi''(x) + \phi'(x) + \lambda x\phi(x) = 0$

   (b) $(1-x^2)\phi''(x) - 2x\phi'(x) = (\lambda - x)\phi(x)$

   (c) $\sin(x)\phi''(x) + \cos(x)\phi'(x) + \lambda\phi(x) = 0.$

3. Compute the *Raleigh coefficient*

$$\frac{\left[-p\phi_n(x)\phi_n'(x)\right]_a^b + \int_a^b p(x)\left[(\phi_n'(x))^2 - q(x)\phi_n^2(x)\right]\,dx}{\int_a^b \phi_n^2(x)\sigma(x)\,dx} \tag{2}$$

   for the special case $p(x) = 1$, $q(x) = 0$ and $\sigma(x) = 1$ and

   (a) $\beta_1 = \beta_3 = 1$, $\beta_2 = \beta_4 = 0$

   (b) $\beta_1 = \beta_3 = 0$, $\beta_2 = \beta_4 = 1$.

4. An operator L acting on a vector space of functions is said to be *self-adjoint* if

$$\int_a^b v(x)\,L(u)(x)\,dx = \int_a^b u(x)\,L(v)(x)\,dx \tag{3}$$

   for all continuously differentiable functions u and v. State whether the following operators are self-adjoint, acting on the space $C^3[a,b]$ of three times continuously differentiable functions *without any boundary condition.* Justify your answers:

(a) $L(f) = f'' + 3f'''$

(b) $L(f) = f^2$

(c) $L(f) = f$