Handout 1 – The method of Characteristics

APM 384 PDEs (Janosch Ortmann)

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1 The Method of Characteristics

In these notes we discuss how to use the method of characteristics to solve first order linear PDEs with two arguments. We will first give the general four-step solution, then justify it and finally go through some worked examples.

The general form of a first-order linear PDE in two arguments is

$$a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} + c(x,y)u(x,y) = g(x,y) \tag{1.1}$$

where $a(x,y), b(x,y), c(x,y), g(x,y)$ are generic functions. Thus, we are given $a(x,y)$, $b(x,y)$, $c(x,y)$ and $g(x,y)$ and we wish to find $u(x,y)$. Consider the *surface of solutions*

$$S = \{(x,y,u(x,y)) \colon x,y \in \mathbb{R}\}. \tag{1.2}$$

The method of characteristics tells us that the general solution to (1.1) is obtained as follows:

Step 1: For any $y_0 \in \mathbb{R}$ solve the ODE initial value problem

$$Y'(t) = \frac{b(t,Y(t))}{a(t,Y(t))} \tag{1.3}$$

$$Y(0) = y_0 \tag{1.4}$$

and denote the solution (which depends on y_0 through the initial condition) by $Y_{y_0}(t)$.

Step 2: For a given point (x,y) find y_0 such that the curve $Y_{y_0}(t)$ goes through (x,y), i.e. solve $Y_{y_0}(x) = y$ for y_0. This gives us an expression $y_0 = p(x,y)$.

Step 3: Solve the ODE $\nu'(t) = \frac{g(t,Y(t)) - c(t,Y(t))\nu(t)}{a(t,Y(t))}$ for ν. The initial condition is an arbitrary function $h(y_0)$ of y_0, in the sense that for each choice of function h we get a solution. Denote the solution (which depends on y_0 through $Y(t)$ and on the function h) by ν_{h,y_0}.

Step 4: The general solution to (1.1) is given by

$$u(x,y) = \nu_{h,p(x,y)}(x).$$

Note that our final answer still depends on the function h. This is because, in essence, a PDE is not enough to determine a solution uniquely. For each 'reasonable' (say, continuously differentiable) function h we get a solution to (1.1). In practice we are usually given some boundary conditions that will further narrow the range of our solutions.

2 Derivation of the method

Let us now see why this is the right thing to do. The essential idea behind the method of characteristics is to characterise S by the curves (called *characteristic curves*) running through it. Our first step is to re-write (1.1) as

$$\begin{pmatrix} a(x,y) \\ b(x,y) \end{pmatrix} \cdot \nabla u = g(x,y) - c(x,y)u(x,y). \tag{2.1}$$

Suppose we have a curve in the solution surface S. Then it must lie in the direction field given by $\begin{pmatrix} a(x,y) \\ b(x,y) \end{pmatrix}$. We will first find the projection $\gamma(t)$ of such curves in the (x,y) plane and then find $\nu(t)$, the position on the u-coordinate.

In order to find γ we make a choice of parametrisation: we choose the x-coordinate to be given by t. Thus we are looking for curves of the form $\gamma(t) = (t, Y(t))$. There is nothing special about this choice, and we could more generally look for curves of the form $(X(t), Y(t))$. However, the former is usually slightly easier and will do for our purposes. We are therefore looking for curves of the form $(t, Y(t))$ whose tangent vector at a point $(t, Y(t))$ is given by $\begin{pmatrix} b(t,Y(t)) \\ a(t,Y(t)) \end{pmatrix}$. This corresponds to the ODE

$$Y'(t) = \frac{b(t,Y(t))}{a(t,Y(t))}. \tag{2.2}$$

We know from the theory of ODEs that for any initial condition $Y(0) = y_0$ we can solve this. Since our answer will depend on the initial condition we denote it by $Y_{y_0}(t)$.

The next step is to find the u-coordinate of our curve, that is we need to find the curve in the solution surface S whose projection is γ. Thus we are looking for the curve $(t, Y(t), \nu(t))$ where $\nu(t) = u(\gamma(t)) = u(t, Y(t))$. By the chain rule and the fact that u solves (1.1),

$$\begin{aligned}
\nu'(t) &= \frac{d}{dt}u(t,Y(t)) = u_x(t,Y(t)) + \frac{b(t,Y(t))}{a(t,Y(t))}u_y(t,Y(t)) \\
&= \frac{1}{a(t,Y(t))}\left[a(t,Y(t))u_x(t,Y(t)) + b(t,Y(t))u_y(t,Y(t))\right] \\
&= \frac{g(t,Y(t)) - c(t,Y(t))u(t,Y(t))}{a(t,Y(t))}
\end{aligned}$$

i.e. ν satisfies the ODE

$$\nu'(t) = \frac{g(t,Y(t)) - c(t,Y(t))\nu(t)}{a(t,Y(t))}$$

Notice that, since $Y(t)$ depends on y_0, so does $\nu(t)$. In order to solve for $\nu(t)$ we need to prescribe an initial condition. However, we have used up all our information. Since the initial condition will also depend on y_0 we simply say that $\nu(0) = h(y_0)$, where h is a general (continuously differentiable) function. Denote this solution by ν_{h,y_0}.

Where did we end up? For any (x, y), the point $(x, y, u(x, y))$ on the surface S must lie on a curve in S, and we have seen that such a curve must be of the form $(t, Y_{y_0}(t), \nu_{h,y_0}(t))$ for some t and some starting point y_0. Of course we need to solve for t and y_0 in terms of x, y, but we have already done this: we know that $y_0 = p(x, y)$ and by comparing the first co-ordinates it's immediately obvious that $t = x$. Thus it follows that

$$u(x, y) = \nu_{h,p(x,y)}(x) \tag{2.3}$$

as claimed above.

3 Examples

Example 3.1. Our first example is the PDE

$$u_x + yu_y + c = 0$$

which corresponds to the choices $a(x, y) = 1$, $b(x, y) = y$, $c(x, y) = 1$ and $g(x, y) = 0$. We follow our four steps, so first we need to solve

$$Y'(t) = Y(t), \qquad Y(0) = y_0$$

which gives us $Y_{y_0}(t) = y_0 e^t$. Thus, solving $y = Y_{y_0}(x) = y$ yields $y_0 = ye^{-x}$. In our notation above, this corresponds to finding that $p(x, y) = ye^{-x}$. Next we need to solve for ν which in this case reduces to

$$\nu'(t) = -\nu(t), \qquad \nu(0) = h(y_0)$$

i.e. we have $\nu_{h,y_0}(t) = h(y_0)e^{-t}$. It follows that the general solution to our problem is

$$u(x, y) = \nu_{h,p(x,y)}(x) = h(ye^{-x})e^{-x}.$$

As discussed above the function h remains undetermined by the PDE. Suppose we additionally impose the boundary condition $u(0, y) = y^2$. Substituting this into our solution yields

$$y^2 = u(0, y) = h(y).$$

Hence the solution to our PDE together with the boundary condition $u(0, y) = y^2$ is given by $u(x, y) = y^2 e^{-3x}$.

Example 3.2. The method of characteristics does not always allow us to give the answer in closed form. Consider

$$u_x + xyu_y = 2y^2.$$

Thus we have $a(x, y) = 1$, $b(x, y) = xy$, $c(x, y) = 0$ and $g(x, y) = 2y^2$. The first step is therefore to find the solution to the initial-value problem

$$Y'(t) = tY(t), \quad Y(0) = y_0.$$

which is given by $Y_{y_0}(t) = y_0 e^{t^2/2}$. Solving $y = Y_{y_0}(x)$ for y_0 yields $y_0 = p(x, y) = ye^{-x^2/2}$. Next we need to solve

$$\nu'(t) = 2Y(t)^2 = 2y_0^2 e^{-t^2}, \qquad \nu(0) = h(y_0).$$

The solution is given by $\nu_{h,y_0}(t) = h(y_0) + 2y_0 e(t)$ where $e(t) = \int_0^t e^{-s^2}$ is the *error function*. Unfortunately the integral cannot be computed in closed form, so this is the best we can do. The general solution is given by

$$u(x, y) = h(ye^{-x^2/2}) + 2ye^{-x^2/2}e(t)$$

where we need to remember that $h(t)$ is an arbitrary function and every choice of $h(t)$ yields a solution whereas $e(t)$ is a fixed, particular function that we know.