Handout 2 – Complex differentiable and harmonic functions

APM 384 PDEs

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1 Complex differentiation

1.1 The complex numbers

Denote by that $\mathbb{C} = \{x + iy\colon x, y \in \mathbb{R}\}$ the field of complex numbers. Here i is the *imaginary unit* and satisfies $i^2 = -1$. Together with commutativity, associativity and distributivity of addition and multiplication this allows us to add, subtract, multiply and divide any two complex numbers: if $x_1, x_2, y_1, y_2 \in \mathbb{R}$,

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2) \tag{1.1}$$

$$(x_1 + iy_1) - (x_2 + iy_2) = (x_1 - x_2) + i(y_1 - y_2) \tag{1.2}$$

$$(x_1 + iy_1) \cdot (x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \tag{1.3}$$

$$\frac{(x_1 + iy_1)}{(x_2 + iy_2)} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\,\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}. \tag{1.4}$$

A number of the form iy where $y \in \mathbb{R}$ is said to be *purely imaginary*. If $x, y \in \mathbb{R}$ then the *real* and *imaginary parts* of $z = x + iy$ are x and y respectively. This is often denoted $x = \Re(z)$ and $y = \Im(z)$. We also denote the *complex conjugate* of z, namely $x - iy$ by $\overline{z}$. Note that

$$\Re(z) = \frac{1}{2}\left(z + \overline{z}\right) \tag{1.5}$$

$$\Im(z) = \frac{1}{2i}\left(z - \overline{z}\right). \tag{1.6}$$

We can also graphically represent $\mathbb{C}$ in the Cartesian plane, denoting $x + iy$ by the point (x, y). This bijection between $\mathbb{R}^2$ and $\mathbb{C}$ will be very useful in the following.

1.2 Complex functions

If $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ is any function, i.e. $f(z) \in \mathbb{C}$ for all $z \in \mathbb{C}$ we can view f as either a single function of one complex variable, or as two functions of two real variables, via the real and imaginary part. We will often denote the latter by u, v, writing

$$u(x, y) = \Re f(x + iy) \quad \text{and} \quad v(x, y) = \Im f(x + iy). \tag{1.7}$$

It is natural to ask about differentiability of complex functions. Since we can divide by complex numbers we can take derivatives of functions with complex arguments, by taking the same definitions as for functions of real variables:

Definition 1.1. A function $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ is said to be *complex differentiable* at $z \in \mathbb{C}$ if the limit, as $h \to 0$ and $h \in \mathbb{C}$ of the difference quotient,

$$\frac{f(z+h) - f(z)}{h} \tag{1.8}$$

exists. If it exists we denote it by $f'(z)$

Notice that we have to let $h \to 0$ through the complex numbers, so we have to be able to approach zero from any direction in the plane. From this fact it follows that complex differentiable functions have very special properties. In particular the real and imaginary part of a complex differentiable function satisfy nice differential equations.

Theorem 1.2 (Cauchy–Riemann equations)**.** *If $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ is complex differentiable and $u(x,y)$ and $v(x,y)$ are defined by* (1.7) *then*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \textit{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{1.9}$$

Proof. Because we can allow $h \to 0$ from any direction, we can in particular let h be real. Thus

$$\begin{aligned} f'(z) &= \lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{f(z+h)-f(z)}{h} = \lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \left[\frac{u(x+h,y)-u(x,y)}{h} + i\frac{v(x+h,y)-v(x,y)}{h}\right] \\ &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}. \end{aligned} \tag{1.10}$$

Similarly, we can let h be purely imaginary, i.e. we write $h = ik$ and let $k \to 0$ throught the reals. Then

$$f'(z) = \lim_{\substack{k\to 0\\ h\in\mathbb{R}}} \frac{f(z+ik)-f(z)}{ik} = \lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \left[\frac{u(x,y+k)-u(x,y)}{ik} + i\frac{v(x+h,y)-v(x,y)}{ik}\right]$$

Since $\frac{1}{i} = -i$ we obtain

$$f'(z) = -iu_y + v_y. \tag{1.11}$$

Since (1.10) and (1.11) refer to the same quantity and two complex numbers are equal if and only if their real and imaginary parts are equal, (1.9) follow. ☐

Example 1.3.

1. We saw in lectures that the function given by $f(z) = z$ is complex differentiable. As an exercise you can check that this is also true for $f(z) = z^n$ where $n \in \mathbb{N}$.

2. Since $\frac{\overline{h}}{h}$ does not have a limit as $h \to 0$ and $h \in \mathbb{C}$, the function given by $f(z) = \overline{z}$ is *not* complex differentiable.

Example 1.4. Another important example of a complex differentiable function is the *exponential function*, defined, for $z \in \mathbb{C}$, by

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

We will take for granted that this series converges for every $z \in \mathbb{C}$, but we will also see that it is complex differentiable. A crucial property is that $\exp(z_1 + z_2) = \exp(z_1)\exp(z_2)$:

$$\begin{aligned}\exp(z_1 + z_2) &= \sum_{n=0}^{\infty} \frac{(z_1+z_2)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} z_1^k z_2^{n-k} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{z_1^k}{k!} \frac{z_2^{n-k}}{(n-k)!}.\end{aligned}$$

Interchanging the two summations (which is justifiable) and introducing the change of variables $m = n - k$ we get

$$\exp(z_1 + z_2) = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{z_1^k}{k!} \frac{z_2^{n-k}}{(n-k)!} = \sum_{k=0}^{\infty} \frac{z_1^k}{k!} \sum_{m=0}^{\infty} \frac{z_2^m}{m!} = \exp(z_1)\exp(z_2).$$

This allows us to prove that exp is complex differentiable:

$$\begin{aligned}\frac{\exp(z+h) - \exp(z)}{h} &= \frac{\exp(h) - 1}{h} \exp(z) = \left[\frac{1}{h} \sum_{n=1}^{\infty} \frac{h^n}{n!} \right] \exp(z) \\ &= \left[1 + \sum_{n=2}^{\infty} \frac{h^{n-1}}{n!} \right] \exp(z)\end{aligned}$$

All the terms in the remaining series in the bracket on the right-hand side are of the form $h^{n-1}n!$ for $n \geq 2$, hence they converge to zero as $h \to 0$. So the complex exponential function is differentiable and has itself as the derivative (just as for the real exponential function).

By comparing the series definition for exp with the Taylor expansions for sin and cos and using the fact that $i^{2n} = (-1)^n$ and $i^{2n+1} = (-1)^n i$ we obtain the following equation, for $x \in \mathbb{R}$

$$e^{iy} = \cos(y) + i\sin(y)$$

and so, more generally,

$$e^{x+iy} = e^x \left(\cos(y) + i\sin(y)\right).$$

Noting that $|e^{iy}| = \sqrt{\cos^2(y) + \sin^2(y)} = 1$, we can write e^{x+iy} as the product of its *magnitude part* e^x, which is a positive number and its *direction part* e^{iy}, which lies on the unit circle (its normal to the circle makes angle y with the positive x-axis).

It also follows that exp is *periodic*, with (complex) period $2\pi i$.

An important result in complex analysis is Cauchy's theorem, which tells us that the values of a complex differentiable function at a point are determined by those on a circle around the point. For $z \in \mathbb{C}$ and $r > 0$ denote by $B(z,r)$ the disk around z or radius r and by $\partial B(z,r)$ the circle around z with radius r:

$$B(z,r) = \{z \in \mathbb{C}\colon |z| < r\} \quad \text{and} \quad \partial B(z,r) = \{z \in \mathbb{C}\colon |z| = r\}.$$

Theorem 1.5 (Cauchy's theorem)**.** *Let f be complex differentiable and $z \in \mathbb{C}$. Then, for any $r > 0$,*

$$f(z) = \frac{1}{2\pi i} \int_{\partial B(0,r)} \frac{f(w)}{z-w}\, dw.$$

One way of proving this theorem is by applying the mean value theorem about harmonic functions, see below.

2 Harmonic Functions

Recall that a function u is said to be *harmonic* in a domain D if it satisfies Laplace's equation there, i.e. $\Delta u(x) = 0$ for all $x \in D$. A surprising fact is that the real and imaginary parts of complex differentiable functions are harmonic:

Theorem 2.1. *Let $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ be complex differentiable and let u, v be defined as in* (1.7) *(i.e. u and v are the real and imaginary parts of f respectively). Then u and v are harmonic.*

Proof. We will show this for u and leave the proof for v as an exercise. By the Cauchy–Riemann equations,

$$\begin{aligned}\Delta u = u_{xx} + u_{yy} &= \frac{\partial}{\partial x}(u_x) + \frac{\partial}{\partial y}(u_y) \\ &= \frac{\partial}{\partial x}(v_y) - \frac{\partial}{\partial y}(v_x) = v_{xy} - v_{yx} = 0.\end{aligned}$$

□

An important property of a harmonic function u is the fact that we can obtain the value of $u(x,y)$ by taking the average value of u on a circle around (x,y). This is referred to as the *mean-value property:*

Theorem 2.2 (Mean-value theorem)**.** *Let u be harmonic in a domain D. If $(x,y) \in D$ then for any $r > 0$ such that $B((x,y),r) \subseteq D$ we have*

$$u(x,y) = \frac{1}{2\pi} \int_0^{2\pi} u(x + r\cos(t), y + r\sin(t))\, dt. \tag{2.1}$$

Remark 2.3. By using the identification between $\mathbb{C}$ and $\mathbb{R}^2$ outlined above and the fact that $e^{it} = \cos(t) + i\sin(t)$ for all $t \in \mathbb{R}$ we could also write (2.1) as follows: for any $z \in D$ such that $B(z,r) \subseteq D$,

$$u(z) = \frac{1}{2\pi} \int_{\partial B(z,r)} u(z + e^{it}) dt. \tag{2.2}$$

Using this together with the fact that the real and imaginary parts of a complex differentiable function are harmonic this suggests an how to prove Cauchy's theorem, Theorem 1.5. But we will not go into details here.

Proof of Theorem 2.2. Let $\phi(r)$ denote the right-hand side. We know that $\phi(0)$ equals the left-hand side. So if we could show that $\phi(r)$ is constant in r (i.e. it does not change if r does) then we would have the result. We know that a function is constant if and only if its derivative is zero, so we need to show that $\phi'(r) = 0$. Interchanging differentiation with the integral,

$$\begin{aligned}\phi'(r) &= \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial}{\partial r} u(x + r\cos(t), y + r\sin(t))dt = \frac{1}{2\pi}\int_0^{2\pi} \begin{pmatrix}\cos(t)\\ \sin(t)\end{pmatrix} \cdot \nabla u \, dt \\ &= \frac{1}{2\pi}\int_{\partial B(z,r)} v \cdot \nabla u \, dt\end{aligned}$$

where $v = \begin{pmatrix}\cos(t)\\ \sin(t)\end{pmatrix}$ is the outward pointing unit normal to our integration curve $\partial B(z, r)$, hence by Green's theorem

$$\phi; (r) = \int_{B(z,r)} \Delta u(x, y)dxdy = 0$$

because u is harmonic. Thus ϕ is constant and hence $\phi(r) = \phi(0) = u(x, y)$ as required. □

Let D be a domain in $\mathbb{R}^2$, i.e. connected (any two points in D can be connected by a curve in D) and open (D does not contain any of its boundary points). We denote the set of boundary points of D by ∂D and the *closure* of D, i.e. the union of D with its boundary points, by $\overline{D}$. Thus $\overline{D} = D \cup \partial D$.

Corollary 2.3 (Maximum principle)**.** *Suppose that u is continuous on $\overline{D}$ and harmonic in D. Then u has its maxima and minima on ∂D. In other words, if u has a maximum or minimum at $(x, y) \in \overline{D}$ then $(x, y) \in \partial D$.*

Proof. This follows from the Mean Value Theorem: if $(x, y) \in D$ then there exists $r > 0$ such that $B((x, y), r) \subseteq D$. But then $u(x, y)$ is the average of the values of $u(x+r\cos(t), y+r\sin(t))$, which cannot be the case if u has a maximum or minimum at (x, y). Thus (x, y) must lie on the boundary. □