Suggested Exercises for the Midterm

APM 384: PDEs

Autumn 2014

1 Questions

Below are some suggested problems you may want to attempt in preparation for the midterm. There are many excellent problems in Haberman for which you can find solutions in the back of the book: those are marked with an asterisk. Of course, the best exercises are those you can find on the various problem sheets. So you should certainly make sure that you understand the solutions to those.

Since you haven't had any exercises on Fourier Series yet you may want to start with exercises 5 onwards going back to exercises 1 to 4.

The answers to these exercises can be found at the end of this sheet. However, there is little to be gained by looking at the answers before seriously attempting to solve the problems by yourself.

1. **The method of characteristics:** In class, the exercises and handouts we saw many examples of how to apply the method of characteristics to fist-order linear PDEs in two variables. Try to extend this method to the homogeneous case in three dimensions: i.e. try to solve the PDE

$$a(x,y,z)u_x + b(x,y,z)u_y + c(x,y,z)u_z + d(x,y,z)u(x,y,z) = 0 \qquad (1)$$

subject to the boundary condition $u(x,y,0) = f(x,y)$.

2. **Separation of Variables:**

    (a) Whenever we solved an eigenvalue problem we chose a particular sign for the eigenvalues. Check that we did not lose any generality by showing that there is no eigenvalue for the other sign.

    (b) Do the starred exercises in section 2.5 of Haberman.

3. **Complex differentiable and harmonic functions:**

    (a) Show that the following functions are harmonic:

$$u_1(x,y) = xy$$
$$u_2(x,y) = e^x \sin(y).$$

(b) Check that the mean value theorem holds by computing

$$\int_0^{2\pi} u_1(x + \cos(\theta), y + \sin(\theta))\, d\theta$$

(c) Find minima and maxima for u_1 on the unit disc $\{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 \leq 1\}$. Check the maximum principle by showing that they lie on the boundary.

4. **Fourier Series:** Show that

   (a) the product of two even functions is even

   (b) the product of two odd functions is even

   (c) the product of an even and an odd function is odd

   (d) the sum of two even functions is even

   (e) the sum of two odd functions is odd

   Is the sum of an even and an odd function even or odd?

5. (a) Find the Fourier series for the following functions $f_j \colon [-\pi, \pi] \longrightarrow \mathbb{R}$

      i. $f_1(x) = x^3$

      ii. $f_2(x) = \cos(x)$

      iii. $f_3(x) = e^x$

   (b) For the examples above check that the method of complex Fourier series works by finding coefficients c_n such that $f_j(x) = \sum_{n \in \mathbb{Z}} c_n e^{i\frac{n\pi x}{L}}$.

6. Recall that product-form solutions to the heat equation on $[0, L]$, with zero temperature prescribed on the boundaries, are given by

$$u_{B,n}(t, x) = B \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 t}$$

   for $B \in \mathbb{R}$ and $n \in \{1, 2, \ldots\}$. Use this, and superposition of solutions, to solve the heat equation with zero temperature on the boundary and initial heat profile given by $u(0, x) = f(x)$ where

   (a) $f(x) = x(x - L)$

   (b) $f(x) = \sin\left(\frac{3\pi x}{L}\right)$

   (c) $f(x) = \cos(\frac{\pi x}{L}) - \cos\left(\frac{3\pi x}{L}\right)$

7. Recall that product-form solutions to the heat equation on $[0, L]$, with perfectly insulated ends, are given by

$$u_{A,n}(t, x) = A \cos\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 t}$$

   for $A \in \mathbb{R}$ and $n \in \{0, 1, 2, \ldots\}$. Use this, and superposition of solutions, to solve the heat equation with insulated ends and initial heat profile given by $u(0, x) = f(x)$ where

(a) $f(x) = 2$

(b) $f(x) = x^2(2x - 3L)$

(c) $f(x) = 2\cos(\frac{3\pi x}{L}) - \cos\left(\frac{5\pi x}{L}\right)$

8. In the lectures we saw that if $f\colon [-L, L] \longrightarrow \mathbb{R}$ is continuously differentiable then we may write $f(x) = \sum_{m\in\mathbb{Z}} c_m e^{\frac{im\pi x}{L}}$. Show that

$$c_m = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-\frac{im\pi x}{L}} \tag{2}$$

for all $m \in \mathbb{Z}$.

2 Answers

1. Suppose u is a solution to the problem and that $X(t)$, $Y(t)$ and $Z(t)$ satisfy the coupled ordinary differential equations

$$\dot{X}(t) = a(X(t), Y(t), Z(t)) \qquad X(0) = x_0 \tag{3}$$
$$\dot{Y}(t) = b(X(t), Y(t), Z(t)) \qquad Y(0) = y_0 \tag{4}$$
$$\dot{Z}(t) = c(X(t), Y(t), Z(t)) \qquad Z(0) = 0 \tag{5}$$

where x_0 and y_0 are two parameters, to be chosen later. Then, by the chain rule and then by the fact that u is a solution, the function $U(t) = u(X(t), Y(t), Z(t))$ satisfies

$$\begin{aligned}
\dot{U}(t) &= \dot{X}(t)\frac{\partial u}{\partial x} + \dot{Y}(t)\frac{\partial u}{\partial y} + \dot{Z}(t)\frac{\partial u}{\partial z}\\
&= a(X(t), Y(t), Z(t))\frac{\partial u}{\partial x} + b(X(t), Y(t), Z(t))\frac{\partial u}{\partial y} + c(X(t), Y(t), Z(t))\frac{\partial u}{\partial z}\\
&= -d(X(t), Y(t), Z(t))u(X(t), Y(t), Z(t)) = -d(X(t), Y(t), Z(t))U(t)\\
U(0) &= u(X(0), Y(0), Z(0)) = u(x_0, y_0, 0) = f(x_0, y_0).
\end{aligned}$$

So in three dimensions we first solve the system (3), (4), (5), then plug in the functions $X(t), Y(t), Z(t)$ into the ODE $\dot{U}(t) = -d(X(t), Y(t), Z(t))U(t)$, which we solve together with the initial condition $U(0) = fx_0, y_0$.

For given (x, y, z) it now remains to choose x_0, y_0 and t such that $X(t) = x$, $Y(t) = y$ and $Z(t) = z$. With these choices, $U(t)$ is then the value of our solution u at the point (x, y, z).

2. See Haberman.

3. (a) Differentiating u_1 twice with respect to either variable gives zero, so the function is harmonic. For u_2 you have two choices. You can either compute directly or use the connection to complex analysis: the real and imaginary parts of a complex differentiable function are harmonic. Since u_2 is the imaginary part of $f(z) = e^z$ it is harmonic.

(b) This is a straightforward computation. For example:

$$\begin{aligned}\int_0^{2\pi} u_1(x+\cos(\theta), y+\sin(\theta))\,d\theta &= \int_0^{2\pi} [(x+\cos(\theta)) - (y+\sin(\theta))]\,d\theta \\ &= 2\pi(x-y) + \int_0^{2\pi} \cos(\theta)\,d\theta - \int_0^{2\pi} \sin(\theta)\,d\theta \\ &= 2\pi(x-y) = 2\pi u(x,y).\end{aligned}$$

(c) u_1 is maximised when $x = 1$ and $y = 0$ and minimised when $x = -1$ and $y = 0$. Thus, the maxima lie on the boundary. Now the maximum of $e^x \cos(y)$ on the unit disc is clearly at $(x, y) = (1, 0)$, since the function e^x is increasing everywhere, and $\cos(y)$ is symmetric about zero and decreasing on the interval $[0, 1]$. Similarly it is easy to verify that the minimum is at $(x, y) = (-1, 0)$.

4. (a) If f, g are even then $(fg)(x) = f(-x)g(-x) = f(x)g(x) = (fg)(x)$, so fg is even.

(b) If f, g are odd then $(fg)(-x) = f(-x)g(-x) = (-f(x))(-g(x)) = (fg)(x)$, so fg is even.

(c) If f is even and g is odd then $(fg)(-x) = f(-x)g(-x) = f(x)(-g(x)) = -(fg)(x)$, so fg is odd.

(d) If f and g are even then $(f+g)(-x) = f(-x)+g(-x) = f(x)+g(x) = (f+g)(x)$, so $f+g$ is even.

(e) If f and g are odd then $(f+g)(-x) = f(-x)+g(-x) = -f(x)-g(x) = -(f+g)(x)$, so $f+g$ is even.

In general the sum of an even and an odd function is neither even nor odd. Take e.g. $f(x) = x$ (odd) and $g(x) = x^2$ (even), then their sum is neither!

5. (a) Since $f(x) = x^3$ defines an odd function it follows immediately that the cosine coefficients satisfy $A_n(f) = 0$. Integrating by parts several times we see that an antiderivative for $\sin(x)x^3$ is given by the function $\frac{x(6-n^2x^2)}{n^3}\cos(n\pi x) + \frac{3(n^2x^2-2)}{n^4}\sin(n\pi x)$. Thus

$$\begin{aligned}B_n(f) &= \frac{1}{\pi}\left[\frac{x(6-n^2x^2)}{n^3}\cos(n\pi x) + \frac{3(n^2x^2-2)}{n^4}\sin(n\pi x)\right]_{x=-\pi}^{\pi} \\ &= \frac{2\pi(n^2\pi^2-6)}{\pi n^3}(-1)^n.\end{aligned}$$

(b) Clearly $A_1(f) = 1$, $A_n(f) = 0$ for $n \neq 1$ and $B_n(f) = 0$ for all n.

(c) We have

$$A_0(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x\,dx = \frac{1}{2\pi}\left[e^x\right]_{x=-\pi}^{\pi} = \frac{e^\pi}{\pi}$$

and, if $n \geq 1$,

$$\begin{aligned} A_n(f) &= \int_{-\pi}^{\pi} \cos(nx)\, e^x \, dx = \left[\frac{e^x}{1+n^2} \, (\cos(nx) + n \sin(nx)) \right]_{x=-\pi}^{\pi} \\ &= \frac{2e^\pi}{1+n^2} \, (-1)^n. \end{aligned}$$

Finally, for $n \geq 1$ we also have

$$\begin{aligned} B_n(f) &= \int_{-\pi}^{\pi} \sin(nx)\, e^x \, dx = \left[\frac{e^x}{1+n^2} \, (\sin(nx) - n \cos(nx)) \right]_{x=-\pi}^{\pi} \\ &= \frac{2ne^\pi}{\pi + \pi n^2} \, (-1)^{n-1}. \end{aligned}$$

6. By superposition of solutions we know that for any real coefficients b_n the function

$$u(t,x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 t}$$

satisfies the PDE and boundary conditions. Hence our task is to find the right coefficients b_n for u to satisfy the initial condition. Since $e^0 = 1$,

$$u(0,x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

it follows by the Fourier theorem that we need to choose $b_n = B_n(f)$, i.e. the n^{th} sine coefficient of the Fourier series of f. I leave you to calculate these for the three initial conditions given.

7. Similar to the question above, except this time we need the cosine coefficients.

8. There are different ways to answer the question; here is one with relatively little computations to do. We already know that the function can be written as such a series. Using this knowledge we only need to compute

$$\begin{aligned} \int_{-L}^{L} e^{\frac{im\pi x}{L}} e^{\frac{in\pi x}{L}} \, dx &= \int_{-L}^{L} e^{\frac{i(m+n)\pi x}{L}} \, dx \\ &= \begin{cases} 2L & \text{if } m+n=0 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Therefore, interchanging infinite summation with integrals without justification as usual,

$$\begin{aligned} \int_{-L}^{L} f(x) e^{\frac{im\pi x}{L}} \, dx &= \sum_{n\in\mathbb{Z}} c_n \int_{-L}^{L} e^{\frac{im\pi x}{L}} \, e^{\frac{in\pi x}{L}} \, dx \\ &= 2Lc_m \end{aligned}$$

from which, of course, the desired identity follows.