Topics in Probability: Problem Sheet 1

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On this first sheet no exercises are assessed. You are encouraged to work through all problems. Unless otherwise specified, $(\Omega, \mathcal{F}, \mathbb{P})$ is a general probability space.

1. Show that for any collection $A_n \in \mathcal{F}, n \in \mathbb{N}$, of measurable sets in Ω there exists *disjoint* $B_n \in \mathcal{F},\ n \in \mathbb{N}$, such that $B_n \subseteq A_n$ for all $n \in \mathbb{N}$ and

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n.$$

2. A set $\mathcal{A}$ of subsets of Ω is called an *algebra* if it contains the empty set and $A, B \in \mathcal{A}$ imply $\Omega \setminus A \in \mathcal{A}$ and $A \cup B \in \mathcal{A}$. (Thus, we have removed the countable part of the union condition.)

   (a) Let $\Omega = \mathbb{N}$ and let $\mathcal{A}$ be the *cofinite algebra* on $\mathbb{N}$, i.e.

   $$\mathcal{A} = \{S \subseteq \mathbb{N} \colon \text{either } S \text{ or } \mathbb{N} \setminus S \text{ is finite}\}.$$

   Show that $\mathcal{A}$ is indeed an algebra, but not a σ-algebra.

   (b) For any set Ω, show that

   $$\mathcal{B} = \{A \subseteq X \colon \text{either } A \text{ or } \Omega \setminus A \text{ is countable}\}$$

   is a σ-algebra on Ω.

   (c) Show that $\mathcal{B} = 2^{\Omega}$ if Ω is countable.

3. Prove Proposition 1.7 from the handout.

4. Show that $X \colon \Omega_1 \longrightarrow \Omega_2$ is $\mathcal{F}_1/\mathcal{F}_2$ measurable if and only if $\sigma(X)$ is contained in $\mathcal{F}_1$.

5. Prove Proposition 1.9 from the handout.

6. Prove that any continuous function between two topological spaces is measurable with respect to their Borel σ-algebras.

7. Let $(X_n, n \in \mathbb{N})$ be non-negative random variables (i.e. measurable maps $X_n \colon \Omega \longrightarrow [0, \infty)$). Show that the following are $[0, \infty]$ valued random variables:

i) $\sup_{n\in\mathbb{N}} X_n$,

ii) $\inf_{n\in\mathbb{N}} X_n$

iii) $\limsup_{n\in\mathbb{N}} X_n$

iv) $\liminf_{n\in\mathbb{N}} X_n$

8. Show that if $X_n\colon \Omega_1 \longrightarrow \Omega_2$ is $\mathcal{F}_1/\mathcal{F}_2$-measurable for each $n \in \mathbb{N}$ and $\mathcal{F}_2 = \sigma(\mathcal{A})$ then

$$\sigma(X_n\colon n \in \mathbb{N}) = \sigma\left\{X_n^{-1}(A)\colon n \in \mathbb{N},\, A \in \mathcal{A}\right\}.$$

9. Let $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ be continous at all but countably many points. Show that f is Borel measurable.

10. Prove Proposition 2.3 from the handout.

11. Prove the *inclusion-exclusion formula*: for $A_1, \ldots, A_n \in \mathcal{F}$,

$$\mathbb{P}\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} \mathbb{P}(A_k) - \sum_{k<l} \mathbb{P}\left(A_k \cap A_l\right) + \sum_{j<k<l} \mathbb{P}\left(A_j \cap A_k \cap A_l\right) - \ldots + (-1)^{n-1}\mathbb{P}\left(\bigcap_{j=1}^{n} A_j\right).$$

(*Hint: Note that* $\bigcup_k A_k = \Omega \setminus \bigcap_{k=1}^{n}(\Omega \setminus A_k)$.)