Topics in Probability: Problem Sheet 2

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On this sheet no exercises are assessed. You are encouraged to work through all problems. Unless otherwise specified, $(\Omega, \mathcal{F}, \mathbb{P})$ is a general probability space.

1. A function $f \colon \mathbb{R} \longrightarrow \mathbb{R}$ is said to be *lower semicontinuous (LSC)* if

$$\liminf_{y \longrightarrow x} f(y) \geq f(x) \qquad \forall\, x \in \mathbb{R}.$$

   Similarly f is said to be *upper semicontinuous (USC)* if $-f$ is lower semicontinuous. Show that upper and lower semicontinuous functions are Borel measurable. (*Hint: what can you say about* $\{x \in \mathbb{R} \colon f(x) \leq a$ *if* f *is LSC?*)

2. (a) Show that our definition of $\int f \mathrm{d}\,\mathbb{P}$ is well defined when f is a simple function. In other words, prove that if

$$f = \sum_{j=1}^{k} c_j 1_{A_j} = \sum_{l=1}^{m} d_l 1_{B_l}$$

   for constants c_j, d_j and measurable sets A_j, B_j, then $\sum_{j=1}^{k} c_j \mathbb{P}(A_j) = \sum_{l=1}^{m} d_l \mathbb{P}(B_l)$.

   (b) Hence, show that $\int (f_1 + f_2) \mathrm{d}\,\mathbb{P} = \int f_1 d\mathbb{P} + \int f_2 \mathrm{d}\,\mathbb{P}$ for all simple functions f_1 and f_2.

3. Let f_1 and f_2 be bounded measurable functions. Show that

$$\int (f_1 + f_2) \mathrm{d}\,\mathbb{P} = \int f_1 d\mathbb{P} + \int f_2 \mathrm{d}\,\mathbb{P}.$$

4. Let $X \colon \Omega \longrightarrow [0, \infty)$ be measurable and such that $\int X \mathrm{d}\,\mathbb{P} = 0$. Prove that $X = 0$ a.s.

5. In this exercise we will prove Jensen's inequality. Let ϕ be a convex function and X a random variable such that $X, \phi(X) \in \mathrm{L}^1(\Omega, \mathcal{F}, \mathbb{P})$.

   (a) Let $c = \mathbb{E}(X)$. Show that there exist $a, b \in \mathbb{R}$ such that if we define $\ell(x) = ax + b$ then $\ell(c) = \phi(c)$ and $\phi(x) \geq \ell(x)$. (*Hint: think about our graphical explanation of what convexity means*).

(b) Use convexity of ϕ to deduce that $\mathbb{E}[\phi(X)] \geq l(\mathbb{E}(X))$ and hence deduce the result.

6. Suppose that $(X_n : n \in \mathbb{N})$ is a sequence of random variables such that $X_n \longrightarrow X$ a.s. Why is X also a random variable? Prove that $X_n \longrightarrow X$ in probability.

7. Use Hölder's inequality to prove *Minkowski's inequality*:

$$\mathbb{E}\left[|X+Y|^p\right] \leq \left(\mathbb{E}\left[|X|^p\right] + \mathbb{E}\left[|Y|^p\right]\right)^p.$$

8. Use Chebychev's inequality to show that if a sequence $(X_n)_{n\in\mathbb{N}}$ of random variables satisfies

   i) $\mathbb{E}[X_n] \longrightarrow a$ as $n \longrightarrow \infty$ for some $a \in \mathbb{R}$
   ii) $\mathbb{E}X_n^2 - (\mathbb{E}[X_n])^2 \longrightarrow 0$ as $n \longrightarrow \infty$

   then $X_n \longrightarrow X$ in probability.

9. Let $(\Omega, \mathcal{F}, \mathbb{P}) = ([0,1], \mathcal{B}[0,1], \mu)$ where μ is Lebesgue measure (uniform measure). Define intervals I_n as follows: for each $n \in \mathbb{N}$ there exist unique $m \in \mathbb{N}$ and $j \in \{0, 1, \dots, 2^m - 1\}$ such that $n = 2^m + j$. Then we define

$$I_{2^m+j} = \left[\frac{j}{2^m}, \frac{j+1}{2^m}\right]$$

   i.e., we set $I_1 = [0, \frac{1}{2}]$, $I_2 = [\frac{1}{2}, 1]$, $I_3 = [0, \frac{1}{4}]$, $I_4 = [\frac{1}{4}, \frac{1}{2}]$ and so on. Define, for $n \in \mathbb{N}$, random variables X_n by $X_n = 1_{I_n}$.

   (a) Show that X_n converges to zero in probability.
   (b) Prove that X_n does not converge almost surely to any random variable.