Topics in Probability: Problem Sheet 3

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On this sheet no exercises are assessed but you are encouraged hand in your solutions to me in class on Monday, September 30. Unless otherwise specified, $(\Omega, \mathcal{F}, \mathbb{P})$ is a general probability space.

1. Let X be a random variable with distribution function F_X. Show that

   i) $\mathbb{P}(X < x) = \lim_{\epsilon \downarrow 0} F(x - \epsilon)$

   ii) $\mathbb{P}(X = x) = F(x) - \lim_{\epsilon \downarrow 0} F(x - \epsilon)$.

2. Let X be a random variable with density function f_x and $\phi \colon \mathbb{R} \longrightarrow \mathbb{R}$ such that $\phi(X) \in \mathrm{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. Show that

$$\mathbb{E}\left[\phi(X)\right] = \int_{\mathbb{R}} \phi(x) f(x) dx.$$

3. i) Let X have standard normal distribution, i.e. its distribution function is given by

$$f_X(x) = \frac{1}{2\pi} e^{-x^2/2}$$

   and recall from lectures that $\mathbb{E}(X) = 0$. Show that $\mathbb{E}[X^2] = 1$, so that $\mathrm{var}(X) = 1$.

   ii) Let $Y = aX + b$. What are $\mathbb{E}(Y)$, $\mathrm{var}(Y)$ and the distribution function of Y?

4. In this exercise you will prove an extension of the Chebyshev inequality.

   i) Let $\phi \colon \mathbb{R} \longrightarrow [0, \infty)$ be measurable and denote, for $A \in \mathbb{B}$, the infimum of ϕ over A by $I_A(\phi)$. Show that

$$\mathbb{P}(X^{-1}(A)) \leq \frac{\mathbb{E}[\phi(X)]}{I_A(\phi)}$$

   ii) Deduce that, for $a > 0$,

$$\mathbb{P}(-a < X < a) \leq \frac{\mathbb{E}\,|X|}{a}$$

5. Let $X_n \in \mathrm{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ such that $\sum_{n=1}^{\infty} \mathbb{E}(X_n) < \infty$. Prove that

$$\mathbb{E}\left[\sum_{n=1}^{\infty}\right] = \sum_{n=1}^{\infty} \mathbb{E}(X_n).$$

6. Let $Y \in \mathrm{L}^2(\Omega, \mathcal{F}, \mathbb{P})$. Show that $\mathbb{E}[Y^2]\,\mathbb{P}(Y > 0) \geq (\mathbb{E}Y)^2$.

7. Let X be a positive integrable random variable. Show that the mapping $\mu\colon \mathcal{F} \longrightarrow \mathbb{R}$ given by

$$\mu(A) = \mathbb{E}(X 1_A) = \int_A X(\omega)\mathbb{P}(d\omega)$$

defines a probability measure on $(\Omega, \mathcal{F}, \mathbb{P})$.

8. Analogously to L^1 and L^2 denote by $\mathrm{L}^k(\Omega, \mathcal{F}, \mathbb{P})$ the space of random variables X such that $|X|^k$ is integrable. Show that $\mathrm{L}^n(\Omega, \mathcal{F}, \mathbb{P}) \subseteq \mathrm{L}^k(\Omega, \mathcal{F}, \mathbb{P})$ whenever $k \leq n$.

9. Recall the following example from lectures: $(\Omega, \mathcal{F}, \mathbb{P}) = ([0,1], \mathcal{B}[0,1], \mu)$ where μ is Lebesgue measure and $X_n = n 1_{[0,1/n]}$. Why do none of our limit theorems apply?