Topics in Probability: Problem Sheet 4

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This is the first exercise sheet that will contribute to your final mark. Please hand in solutions before the lecture on **Wednesday October 9**. Unless otherwise specified $(\Omega, \mathcal{F}, \mathbb{P})$ is a generic probability space. Exercises 1 is not for credit, but you are still enouraged to hand in your solutions to test your understanding. Question 2 is worth 20 points, Exercises 3-11 are worth 10 points (so there are 100 points available in total).

1. (Not for credit.) Prove that if $\mathcal{A}_1, \ldots, \mathcal{A}_n \subseteq \mathcal{F}$ are independent and $A_j \in \mathcal{A}_j$ for $j > 1$ then

$$\mathcal{L} = \left\{ A \in \mathcal{F} \colon \mathbb{P}\left(A \cap \bigcap_{j=2}^{k} A_j \right) = \mathbb{P}(A)\mathbb{P}\left(\bigcap_{j=2}^{k} A_j \right) \right\}$$

is a λ-system.

2. Find four random variables taking values in $\{0, 1\}$ that are not independent, but such that any three of them are independent.

3. Let $(\Omega, \mathcal{F}, \mathbb{P}) = ([0,1], \mathcal{B}[0,1], \lambda)$ where λ denotes Lebesgue measure. Show that the random variables $X_1, \ldots, X_N$ defined by $X_n(\omega) = \sin(2n\pi\omega)$ are uncorrelated but not independent.

4. If $F_1, F_2 \colon \mathbb{R} \longrightarrow \mathbb{R}$ are Borel measurable functions and μ_1, μ_2 are probability measures on $(\mathbb{R}, \mathbb{B})$ then we define their convolutions by

$$F_1 * F_2(z) = \int_{\mathbb{R}} F_1(z-y) F_2(y) \,\mathrm{d}\, y; \qquad z \in \mathbb{R}$$

$$F_1 * \mu_2(z) = \int_{\mathbb{R}} F_1(z-y) \mu_2(\mathrm{d}\, y); \qquad z \in \mathbb{R}$$

$$(\mu_1 * \mu_2)(A) = \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbf{1}_A(z-y) \mu_1(\mathrm{d}\, z) \mu_2(\mathrm{d}\, y) \qquad A \in \mathbb{B}.$$

   (a) Show that if $\mu_2(A) = \int_A F_2(x) dx$ for all $A \in \mathbb{B}$ then $F_1 * \mu_2 = F_1 * F_2$.

   For parts (b) and (c) let X and Y be independent random variables with probability laws μ_X, μ_Y and probability distribution functions F_X and F_Y respectively.

(b) Show that the law of $X+Y$ is given by $\mu_X * \mu_Y$.

(c) Suppose now that the law of Y is absolutely continuous with respect to Lebesgue measure, with density f_Y, so that $\mu_Y(A) = \int_A f_Y(y)dy$. Show that the probability distribution function of $X+Y$ is given by $F_X * f_Y$.

5. Let X, Y be independent random variables with probability laws $\sum_{n=1}^{\infty} p_n \delta_n$ and $\sum_{n=1}^{\infty} q_n \delta_n$ respectively (of course $p_n, q_n \geq 0$ for all n and $\sum_n p_n = \sum_n q_n = 1$). State and prove a formula for the probability law of $X+Y$.

6. Let X and Y be independent random variables with Poisson distribution with mean m_X and m_Y respectively. What is the distribution of $X+Y$?

7. Let X, Y be independent. Show that $\text{var}(X+Y) = \text{var}(X) + \text{var}(Y)$.

8. Let X, Y be independent random variables with distribution functions F_X, F_Y and probability laws μ_X, μ_Y respectively. Show that

$$\mathbb{P}\{a < X \leq Y \leq b\} = \int_{(a,b]} (F_X(y) - F_X(a))\, \mu_Y(dy).$$

9. Show that if X is a random variable that is independent of itself, it must be constant. What values can $\mathbb{P}(A)$ take if $A \in \mathcal{F}$ is independent of itself?

10. Suppose that X, Y are non-negative, independent random variables with distribution functions F_X and F_Y respectively. State and prove a formula for the distribution function of XY.